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Fundamental Study

Positive varieties of tree languages

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Abstract

Pin's variety theorem for positive varieties of string languages and varieties of finite ordered semigroups is proved for trees, i.e., a bijective correspondence between positive varieties of tree languages and varieties of finite ordered algebras is established. This, in turn, is extended to generalized varieties of finite ordered algebras, which corresponds to Steinby's generalized variety theorem. Also, families of tree languages and classes of ordered algebras that are definable by ordered (syntactic or translation) monoids are characterized.

Keywords: Tree languages; Tree automata; Variety theorem; Ordered algebras; Ordered monoids

Contents

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1. Introduction

The story of variety theory begins with Eilenberg's celebrated variety theorem [5] which was motivated by characterizations of several families of string languages by syntactic monoids or semigroups (see [5,12]), above all by Schützenberger's [19] theorem connecting star-free languages and aperiodic monoids. A fascinating feature of this variety theorem is the existence of its many instances. Indeed, most of interesting classes of algebraic structures form varieties, and similarly, most of interesting families of tree or string languages in the literature turn out to be varieties of some kind.

Eilenberg's theorem has since been extended in various directions. One of these extensions, which is generalized in this paper for trees, is Pin's positive variety theorem [13] which established a bijective correspondence between positive varieties of string languages and varieties of ordered semigroups. Another extension is Thérien's [24] which includes also varieties of congruences on free monoids.

Concerning trees, which are studied on the level of universal algebra, Steinby's variety theorem [21] for varieties of tree languages and varieties of finite algebras was the first one of this kind. The correspondence with varieties of congruences, and some other generalizations, were added later by Almeida [1] and Steinby [22,23]. Another variety theorem for trees is Ésik's [6] correspondence between families of tree languages and theories (see also [7]).

As Ésik [6] notes any variety theorem connects families of tree languages with classes of some structures via their "syntactic structures". One of these syntactic structures is the syntactic semigroup, or monoid, of a tree language introduced by Thomas [25] and further studied by Salomaa [18]. A different formalism, based on essentially the same concept, was brought up by Nivat and Podelski [10,15]. Very recently a variety theorem for syntactic semigroups, or monoids, was proved by Salehi [16]. The newest syntactic structure for binary trees is the syntactic tree algebra introduced by Wilke [27] for which a variety theorem is proved by Salehi and Steinby [17].

In Section 2, we review basic notions of ordered algebras, ideals and quotient algebras. Ordered algebras play an important role in the field, as Bloom and Wright [4] put it

“Ever since Scott popularized their use in [20], ordered algebras have been used in many places in theoretical computer science’’.

In Section 3, positive varieties of tree languages are introduced and a variety theorem for these varieties and varieties of finite ordered algebras is proved. Informally speaking, a positive variety is a family of recognizable languages which satisfies the definition of a variety except for being closed under complements. Several families of (tree or string) languages are known to be closed under all the variety operations, including intersections and unions, but not under complementation. Pin’s positive variety theorem [13] provides a characterization for these families via ordered semigroups, see also [8,14].

In Section 4, positive variety theorem from Section 3 is extended to generalized varieties. Generalized varieties were introduced by Steinby [23], where generalization refers to omitting the condition of having a fixed ranked alphabet. Namely, a generalized variety of tree languages or of finite algebras contains tree languages or algebras over any ranked alphabet. This is used for proving a variety theorem for trees and ordered monoids in Section 5.

In Section 5, the results of Salehi [16] are extended to ordered monoids. Roughly speaking, a triple correspondence between generalized varieties of finite ordered algebras, generalized positive varieties of tree languages and varieties of finite ordered monoids is presented. This suggests the thesis that once the condition of being closed under complements is removed from the definition of variety, the resulted family, called positive variety, corresponds to a class of ordered syntactic structures of the variety; see also the positive variety theorem by Ésik in [6, Section 12].

Throughout the paper some examples are presented for illustrating the theories and their applicabilities. They are motivated by the string case examples from [13]. Although the obtained correspondences are expected, the tree case appears to be technically more difficult than the string case.

At the end of the paper, Index of notation is provided for readers’ convenience.

2. Ordered algebras

In this section, after reviewing the terminology of ordered sets and ordered algebras, we define the notions of ideals, quotient ordered algebras and syntactic ordered algebras; see also [3,26].

2.1. Basic notions

Let A be a set. The diagonal relation on A is denoted by Δ_A. For binary relations δ and σ on A, the *inverse* of δ and the composition of δ and σ are denoted by δ^{-1} and $\delta \circ \sigma$, respectively. For an equivalence relation θ on A, the equivalence θ-class of an $a \in A$ is $a/\theta = \{b \in A \mid a\theta b\}$ and the quotient set A/θ is $\{a/\theta \mid a \in A\}$.

It is easy to see that for a quasi-order (i.e. a reflexive and transitive binary relation) $\preccurlyeq$ on A, the relation $\theta = \preccurlyeq \cap \preccurlyeq^{-1}$ is an equivalence relation on A, called the *equivalence relation of* $\preccurlyeq$, and the relation $\leqslant$ defined on the quotient set A/θ by $a/\theta \leqslant b/\theta \iff a \preccurlyeq b$ for $a, b \in A$, is a well-defined order on A/θ. This order $\leqslant$ on A/θ is called the *order induced by the quasi-order* $\preccurlyeq$ on A.

A finite set of function symbols is called a *ranked alphabet*. If Σ is a ranked alphabet, the set of m-ary function symbols of Σ is denoted by Σ_m ($m \geqslant 0$). In particular, Σ_0 is the set of constant symbols of Σ. For a ranked alphabet Σ, a Σ-*algebra* is a structure $\mathcal{A} = (A, \Sigma)$ where A is a set, and the operations of Σ are interpreted in A, that is to say, any $c \in \Sigma_0$ is interpreted by an element $c^{\mathcal{A}} \in A$, and any $f \in \Sigma_m$ ($m > 0$) is interpreted by an m-ary function $f^{\mathcal{A}} : A^m \to A$.

An equivalence relation θ on A is a Σ-*congruence* on $\mathcal{A}$ if for any $f \in \Sigma_m$ and $a_1, \ldots, a_m, b_1, \ldots, b_m \in A$, the relation $f^{\mathcal{A}}(a_1, \ldots, a_m)\, \theta\, f^{\mathcal{A}}(b_1, \ldots, b_m)$ holds whenever $a_1\, \theta\, b_1, \ldots, a_m\, \theta\, b_m$.

Let Σ be a ranked alphabet. An *ordered* Σ-*algebra* is a structure $\mathcal{A} = (A, \Sigma, \leqslant)$ where the structure (A, Σ) is an algebra and $\leqslant$ is an order on A which satisfies the following property: for any $f \in \Sigma_m$ ($m > 0$) and $a_1, \ldots, a_m, b_1, \ldots, b_m \in A$, if $a_1 \leqslant b_1, \ldots, a_m \leqslant b_m$ then $f^{\mathcal{A}}(a_1, \ldots, a_m) \leqslant f^{\mathcal{A}}(b_1, \ldots, b_m)$; cf. [26, Section 4.2.1]. We note that any algebra (A, Σ) in the classical sense is an ordered algebra (A, Σ, Δ_A) in which the order relation is equality.

Let $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Sigma, \leqslant')$ be two ordered algebras.

The structure $\mathcal{B}$ is an *order subalgebra* of $\mathcal{A}$, in notation $\mathcal{B} \subseteq \mathcal{A}$, if (B, Σ) is a subalgebra of (A, Σ) and $\leqslant'$ is the restriction of $\leqslant$ to B.

A mapping $\varphi : A \to B$ is an *order morphism* if it is a Σ-morphism, i.e., $c^{\mathcal{A}}\varphi = c^{\mathcal{B}}$ and $f^{\mathcal{A}}(a_1, \ldots, a_m)\varphi = f^{\mathcal{B}}(a_1\varphi, \ldots, a_m\varphi)$ for any $c \in \Sigma_0$, $f \in \Sigma_m$ ($m > 0$), and $a_1, \ldots, a_m \in A$, and preserves the orders, i.e., for any $a, b \in A$ if $a \leqslant b$ then $a\varphi \leqslant' b\varphi$. In that case we write $\varphi : \mathcal{A} \to \mathcal{B}$. The order morphism φ is an *order epimorphism* if it is surjective, and then $\mathcal{B}$ is an *order epimorphic image of* $\mathcal{A}$, in notation $\mathcal{B} \leftarrow \mathcal{A}$. If $\mathcal{B}$ is an order epimorphic image of an order subalgebra of $\mathcal{A}$, then $\mathcal{B}$ is said to *divide* $\mathcal{A}$, in notation $\mathcal{B} \prec \mathcal{A}$. If φ is injective then it is an *order monomorphism*. When φ is bijective and its inverse is also an order morphism, then it is an *order isomorphism*. We write $\mathcal{A} \cong \mathcal{B}$ when $\mathcal{A}$ and $\mathcal{B}$ are order isomorphic.

The direct product of $\mathcal{A}$ and $\mathcal{B}$ is the structure $(A \times B, \Sigma, \leqslant\times\leqslant')$ where $(A \times B, \Sigma)$ is the product of the algebras (A, Σ) and (B, Σ), and the relation $\leqslant\times\leqslant'$ is defined on $A \times B$ by $(a, b) \leqslant\times\leqslant' (c, d) \iff a \leqslant c \;\&\; b \leqslant' d$ for $(a, b), (c, d) \in A \times B$. It is easy to see that the structure $(A \times B, \Sigma, \leqslant\times\leqslant')$ is an ordered algebra which is denoted by $\mathcal{A} \times \mathcal{B}$.

A *variety of finite ordered algebras*, abbreviated by VFOA, is a class of finite ordered algebras closed under order subalgebras, order epimorphic images, and direct products.

2.2. Ideals and quotient ordered algebras

Let $\mathcal{A} = (A, \Sigma, \leqslant)$ be an ordered algebra.

Definition 2.1. A *quasi-order on* $\mathcal{A}$ is a quasi-order $\preccurlyeq$ on A that contains $\leqslant$, i.e., $\preccurlyeq\; \supseteq\; \leqslant$, and is compatible with Σ, i.e., for any $f \in \Sigma_m$ ($m > 0$) and $a_1, \ldots, a_m, b_1, \ldots, b_m \in A$, $f^{\mathcal{A}}(a_1, \ldots, a_m) \preccurlyeq f^{\mathcal{A}}(b_1, \ldots, b_m)$ holds whenever $a_1 \preccurlyeq b_1, \ldots, a_m \preccurlyeq b_m$.

Definition 2.2. For a quasi-order $\preccurlyeq$ on $\mathcal{A}$, the *quotient* of $\mathcal{A}$ under $\preccurlyeq$ is the structure $\mathcal{A}/\!\preccurlyeq\; = (A/\theta, \Sigma, \leqslant)$ where $\theta = \;\preccurlyeq \cap \preccurlyeq^{-1}$ is the Σ-congruence induced by $\preccurlyeq$ and $\leqslant$ is the order induced by $\preccurlyeq$; cf. [26].

For sets A and B and a mapping $\phi : A \to B$, if ρ is a relation on B then $\phi \circ \rho \circ \phi^{-1}$ is a relation on A determined by

$$a \ \varphi \circ \rho \circ \varphi^{-1} \ b \iff (a\varphi)\, \rho\, (b\varphi).$$

Proposition 2.3. *For ordered algebras $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Sigma, \leqslant')$ and order morphism $\varphi : \mathcal{A} \to \mathcal{B}$, if $\preccurlyeq$ is a quasi-order on $\mathcal{B}$ then $\varphi \circ \preccurlyeq \circ \varphi^{-1}$ is a quasi-order on $\mathcal{A}$. Moreover, the following hold*:

(1) *the image of $\mathcal{A}$, $\mathcal{A}\varphi = (A\varphi, \Sigma, \leqslant'')$ where $\leqslant''$ is the restriction of $\leqslant'$ on $A\varphi$, is an order subalgebra of $\mathcal{B}$*,
(2) *$\mathcal{A}/\varphi \circ \preccurlyeq \circ \varphi^{-1} \cong \mathcal{A}\varphi/\preccurlyeq'$ where $\preccurlyeq'$ is the restriction of $\preccurlyeq$ on $A\varphi$, and*
(3) *if φ is an order epimorphism then $\mathcal{A}/\varphi \circ \preccurlyeq \circ \varphi^{-1} \cong \mathcal{B}/\preccurlyeq$.*

Proof. The fact that $\varphi \circ \preccurlyeq \circ \varphi^{-1}$ is a quasi-order on $\mathcal{A}$ and statement (1) are straightforward, and (3) follows from (2). For proving (2) we note that the mapping $\psi : \mathcal{A}/\varphi \circ \preccurlyeq \circ \varphi^{-1} \to \mathcal{A}\varphi/\preccurlyeq'$ defined by $(a/\varphi \circ \theta \circ \varphi^{-1})\psi = a\varphi/\theta$ for $a \in A$, where $\theta = \ \preccurlyeq \cap \preccurlyeq^{-1}$, is an order isomorphism. $\square$

The particular case of the Proposition 2.3 when $\preccurlyeq \ = \ \leqslant'$ is of interest: then $\theta = \Delta_B$ and $\varphi \circ \theta \circ \varphi^{-1} = \ker\varphi$, and hence we get the first homomorphism theorem for ordered algebras, i.e., $\mathcal{A}/\varphi \circ \leqslant' \circ \varphi^{-1} \cong \mathcal{A}\varphi$, see [26]. Results similar to Proposition 2.3 for semigroups can be found in [9].

Proposition 2.4. *Let $\mathcal{A} = (A, \Sigma, \leqslant)$ be an ordered algebra, and $\preccurlyeq, \preccurlyeq'$ be two quasi-orders on $\mathcal{A}$.*
(1) *If $\preccurlyeq \ \subseteq \ \preccurlyeq'$ then $\mathcal{A}/\preccurlyeq' \ \leftarrow \ \mathcal{A}/\preccurlyeq$.*
(2) *The relation $\preccurlyeq \cap \preccurlyeq'$ is a quasi-order on $\mathcal{A}$ and $\mathcal{A}/\preccurlyeq \cap \preccurlyeq'$ is an order subalgebra of $\mathcal{A}/\preccurlyeq \times \mathcal{A}/\preccurlyeq'$.*

The proof is straightforward.

Let us recall the definition of translations of an algebra (see e.g. [21–23]). For an algebra $\mathcal{A} = (A, \Sigma)$, an m-ary function symbol $f \in \Sigma_m$ $(m > 0)$ and elements $a_1, \ldots, a_m \in A$, the term $f^{\mathcal{A}}(a_1, \ldots, \xi, \ldots, a_m)$ where the new symbol ξ sits in the ith position, for some $i \leqslant m$, determines a unary function $A \to A$ defined by $a \mapsto f^{\mathcal{A}}(a_1, \ldots, a, \ldots, a_m)$ which is an *elementary translation* of $\mathcal{A}$. The set of translations of $\mathcal{A}$, denoted by $\mathrm{Tr}(\mathcal{A})$, is the smallest set that contains the identity function and elementary translations and is closed under composition of unary functions. The composition of translations p and q is denoted by $q \cdot p$, that is $(q \cdot p)(a) = p(q(a))$ for any $a \in A$. The set $\mathrm{Tr}(\mathcal{A})$ equipped with the composition operation is a monoid, called the *translation monoid* of $\mathcal{A}$.

Definition 2.5. An *ideal* of $\mathcal{A} = (A, \Sigma, \leqslant)$, in notation $I \trianglelefteq A$, is a subset $I \subseteq A$ such that $a \leqslant b \in I$ implies $a \in I$ for every $a, b \in A$. For any $a \in A$, $(a] = \{b \in A \mid b \leqslant a\}$ is the ideal of $\mathcal{A}$ *generated* by a.

The *syntactic quasi-order* of an ideal $I \trianglelefteq \mathcal{A}$, denoted by $\preccurlyeq_I$, is defined by

$$a \preccurlyeq_I b \iff \big(\forall p \in \mathrm{Tr}(\mathcal{A})\big)\big(p(b) \in I \ \Rightarrow \ p(a) \in I\big)$$

for $a, b \in A$. The *syntactic ordered algebra* of I is the quotient ordered algebra $\mathrm{SOA}(I) = \mathcal{A}/\preccurlyeq_I$, also denoted by $\mathcal{A}/I$ (cf. [13]).

We note that for any ideal I the equivalence relation θ_I of $\preccurlyeq_I$ is the syntactic congruence of I in the classical sense (see e.g. [21,22]):

$$a\,\theta_I\,b \iff (\forall p \in \mathrm{Tr}(\mathcal{A}))(p(a) \in I \Leftrightarrow p(b) \in I).$$

It is known that the syntactic congruence of I is the greatest congruence that saturates I [21,22]. Correspondingly, the syntactic quasi-order of I is the greatest quasi-order on $\mathcal{A}$ that satisfies $a \preccurlyeq b \in I \Rightarrow a \in I$ for all $a, b \in A$.

Trivially, any subset $I \subseteq A$ of the ordered algebra $\mathcal{A} = (A, \Sigma, \Delta_A)$ is an ideal of $\mathcal{A}$. The following is essentially Lemma 3.2 of Steinby [22].

Proposition 2.6. *Let $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Sigma, \leqslant')$ be two ordered algebras, and $\varphi : \mathcal{A} \to \mathcal{B}$ be an order morphism. The mapping φ induces a monoid morphism $\mathrm{Tr}(\mathcal{A}) \to \mathrm{Tr}(\mathcal{B})$, $p \mapsto p_\varphi$, such that $p(a)\varphi = p_\varphi(a\varphi)$ for any $a \in A$. Moreover, if φ is an order epimorphism then the induced mapping is a monoid epimorphism.*

For a subset $D \subseteq A$ and a translation $p \in \mathrm{Tr}(\mathcal{A})$, the inverse translation of D under p is $p^{-1}(D) = \{a \in A \mid p(a) \in D\}$, and for an order morphism $\varphi : \mathcal{B} \to \mathcal{A}$, the inverse image of D under φ is $D\varphi^{-1} = \{b \in B \mid b\varphi \in D\}$.

Positive Boolean operations are intersection and union of sets, while Boolean operations also include the complement operation. It can be easily proved that for ordered algebras $\mathcal{A}$ and $\mathcal{B}$, ideals $I, J \trianglelefteq \mathcal{A}$, $K \trianglelefteq \mathcal{B}$, and order morphism $\varphi : \mathcal{A} \to \mathcal{B}$, the sets $I \cap J$, $I \cup J$, $p^{-1}(I)$ and $K\varphi^{-1}$ are ideals of $\mathcal{A}$. This is formulated in the following lemma whose proof is straightforward (cf. [13]). Note that the complement of an ideal is not necessarily an ideal.

Lemma 2.7. *The collection of all ideals of any ordered algebra is closed under positive Boolean operations, inverse translations and inverse order morphisms.*

Proposition 2.8. *Let $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Sigma, \leqslant')$ be ordered algebras and $I, J \trianglelefteq \mathcal{A}$, $K \trianglelefteq \mathcal{B}$ be ideals. Then the following inclusions hold:*

(1) $\preccurlyeq_{I\cap J}, \preccurlyeq_{I\cup J} \supseteq \preccurlyeq_I \cap \preccurlyeq_J$;

(2) $\preccurlyeq_{p^{-1}(I)} \supseteq \preccurlyeq_I$ *for any* $p \in \mathrm{Tr}(\mathcal{A})$;

(3) $\preccurlyeq_{K\varphi^{-1}} \supseteq \varphi \circ \preccurlyeq_K \circ \varphi^{-1}$ *for any order morphism* $\varphi : \mathcal{A} \to \mathcal{B}$, *and* $\preccurlyeq_{K\varphi^{-1}} = \varphi \circ \preccurlyeq_K \circ \varphi^{-1}$ *if φ is an order epimorphism.*

Proof. Statements (1) and (2) are obvious. We prove (3). Assume $(a, b) \in \varphi \circ \preccurlyeq_K \circ \varphi^{-1}$ for some $a, b \in A$. Then $a\varphi \preccurlyeq_K b\varphi$. Hence, for any $p \in \mathrm{Tr}(\mathcal{A})$, if $p(b) \in K\varphi^{-1}$ then $p(b)\varphi \in K$, what means $p_\varphi(b\varphi) \in K$. This implies now $p_\varphi(a\varphi) \in K$, i.e., $p(a)\varphi \in K$, and so $p(a) \in K\varphi^{-1}$.

Therefore $a \preccurlyeq_{K\varphi^{-1}} b$, and hence $\varphi \circ \preccurlyeq_K \circ \varphi^{-1} \subseteq \preccurlyeq_{K\varphi^{-1}}$. In the case when φ is surjective we note that, by Proposition 2.6, every translation $q \in \mathrm{Tr}(\mathcal{B})$ is of the form p_φ for some

$p \in \mathrm{Tr}(\mathcal{A})$. Thus in this case $\preccurlyeq_{K\varphi^{-1}} \subseteq \varphi \circ \preccurlyeq_K \circ \varphi^{-1}$ holds, and so does the equality $\preccurlyeq_{K\varphi^{-1}} = \varphi \circ \preccurlyeq_K \circ \varphi^{-1}$. $\square$

Combining Propositions 2.8, 2.4 and 2.3 we get the following.

Corollary 2.9. *For ordered algebras* $\mathcal{A} = (A, \Sigma, \leqslant)$ *and* $\mathcal{B} = (B, \Sigma, \leqslant')$, *ideals* $I, J \trianglelefteq \mathcal{A}$ *and* $K \trianglelefteq \mathcal{B}$, *translation* $p \in \mathrm{Tr}(\mathcal{A})$ *and order morphism* $\varphi : \mathcal{A} \to \mathcal{B}$,
(1) $\mathrm{SOA}(I \cap J), \mathrm{SOA}(I \cup J) \prec \mathrm{SOA}(I) \times \mathrm{SOA}(J)$;
(2) $\mathrm{SOA}(p^{-1}(I)) \leftarrow \mathrm{SOA}(I)$;
(3) $\mathrm{SOA}(K\varphi^{-1}) \prec \mathrm{SOA}(K)$ *and if* φ *is an order epimorphism then* $\mathrm{SOA}(K\varphi^{-1}) \cong \mathrm{SOA}(K)$.

2.3. Examples

For an algebra $\mathcal{A} = (A, \Sigma)$, the set of *non-trivial* translations $\mathrm{TrS}(\mathcal{A})$ of $\mathcal{A}$ consists of elementary translations $f^{\mathcal{A}}(a_1, \ldots, \xi, \ldots, a_m)$ for any $f \in \Sigma_m$ and $a_1, \ldots, a_m \in A$, and their compositions. We note that $\mathrm{TrS}(\mathcal{A})$ does not automatically include the identity translation 1_A. The set $\mathrm{TrS}(\mathcal{A})$ with the composition operation is a semigroup, called the *translation semigroup* of $\mathcal{A}$.

2.3.1. Ordered nilpotent algebras

Definition 2.10. An ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$ is *ordered n-nilpotent* ($n \in \mathbb{N}$) if $p_1 \cdots p_n(a) \leqslant b$ holds for all $a, b \in A$ and non-trivial translations $p_1, \ldots, p_n \in \mathrm{TrS}(\mathcal{A})$. An ordered algebra is *ordered nilpotent* if it is ordered n-nilpotent for some $n \in \mathbb{N}$. The class of all ordered nilpotent Σ-algebras is denoted by $\mathbf{Nil}(\Sigma)$.

An element $a_0 \in A$ is a *trap* of $\mathcal{A}$ if $p(a_0) = a_0$ holds for any $p \in \mathrm{Tr}(\mathcal{A})$.

Lemma 2.11. *Every ordered n-nilpotent algebra* $\mathcal{A} = (A, \Sigma, \leqslant)$ *has a unique trap which is the least element of the algebra.*

Proof. Clearly $p_1 \cdots p_n(a) \leqslant q_1 \cdots q_n(b) \leqslant p_1 \cdots p_n(a)$ holds for all non-trivial translations $p_1, \ldots, p_n, q_1, \ldots, q_n \in \mathrm{TrS}(\mathcal{A})$ and $a, b \in A$. Thus $p_1 \cdots p_n(a) = q_1 \cdots q_n(b)$ and let a_0 be this element. Then $p(a_0) = a_0$ and $a_0 \leqslant a$ for any $p \in \mathrm{TrS}(\mathcal{A})$ and $a \in A$. Therefore, a_0 is the unique trap of $\mathcal{A}$ and it is the least element. $\square$

Proposition 2.12. *Class* $\mathbf{Nil}(\Sigma)$ *of all ordered nilpotent* Σ*-algebras is a variety of finite ordered algebras.*

Proof. It can be easily seen that the class of ordered n-nilpotent algebras is closed under order subalgebras and direct products. To see that it is closed under order epimorphic images, let $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Sigma, \leqslant')$ be two ordered algebras, such that $\mathcal{A}$ is an ordered n-nilpotent algebra and let $\varphi : \mathcal{A} \to \mathcal{B}$ be an order epimorphism. Let $b, d \in B$ be two elements and $q_1, \ldots, q_n \in \mathrm{TrS}(\mathcal{B})$ be non-trivial translations. There are $a, c \in A$, such that $b = a\varphi$ and $d = c\varphi$, and by Proposition 2.6, there are $p_1, \ldots, p_n \in \mathrm{TrS}(\mathcal{A})$ such that $(p_j)_\varphi = q_j$ for every $j \leqslant n$. From $p_1 \cdots p_n(a) \leqslant c$, the inequality $p_1 \cdots p_n(a)\varphi \leqslant' c\varphi$ follows and this implies $(p_1)_\varphi \cdots (p_n)_\varphi(a\varphi) \leqslant' c\varphi$. Thus $q_1 \cdots q_n(b) \leqslant' d$ holds. Hence, $\mathcal{B}$ is an ordered n-nilpotent algebra.

Finally, the claim follows from the fact that an ordered n-nilpotent algebra is an ordered $(n+1)$-nilpotent algebra as well. $\square$

2.3.2. Semilattice algebras and symbolic ordered algebras

Finite sequences of elements of a set D are displayed in the bold face, for example $\mathbf{d}$ is a (possibly empty) sequence $\langle d_1, \ldots, d_m \rangle$ where $d_1, \ldots, d_m$ are all members of D. For simplicity we write $\mathbf{d} \in D$ when all components of the sequence $\mathbf{d}$ belong to D. In that case for a function symbol $f \in \Sigma_{m+1}$, $f(d, \mathbf{d})$ stands for $f(d, d_1, \ldots, d_m)$.

Definition 2.13. An algebra $\mathcal{A} = (A, \Sigma)$ is a *semilattice algebra* if it satisfies the following two identities for any $f, g \in \Sigma$ and $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, a \in A$:

$$f^{\mathcal{A}}(\mathbf{a}, f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}), \mathbf{b}) = f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b});$$

$$f^{\mathcal{A}}(\mathbf{a}, g^{\mathcal{A}}(\mathbf{c}, a, \mathbf{d}), \mathbf{b}) = g^{\mathcal{A}}(\mathbf{c}, f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}), \mathbf{d}).$$

A monoid $(M, \cdot)$ is a *semilattice monoid* if it is commutative and idempotent, i.e., $a \cdot a = a$ and $a \cdot b = b \cdot a$ for any $a, b \in M$.

Lemma 2.14. *An algebra is semilattice if and only if its translation monoid is semilattice.*

Lemma 2.15. *Let $\mathcal{A} = (A, \Sigma)$ be a semilattice algebra. For $a, b \in A$ and translations $p, q \in \mathrm{Tr}(\mathcal{A})$ the following hold*:

(1) *if $p(q(a)) = a$ then $p(a) = q(a) = a$;*
(2) *if $p(a) = b$ and $a = q(b)$ then $a = b$.*

Proof. The claim (2) is an immediate corollary of (1). Let us prove (1). Suppose $p, q \in \mathrm{Tr}(\mathcal{A})$. Since $q \cdot q = q$, $p \cdot p = p$ and $q \cdot p = p \cdot q$, we have $q(a) = q(p(q(a))) = q(q(p(a))) = q(p(a)) = p(q(a)) = a$, and similarly $p(a) = p(p(q(a))) = p(q(a)) = a$.

Lemma 2.16. *Let $\mathcal{A} = (A, \Sigma)$ be a semilattice algebra. For $f, g \in \Sigma$ and $\mathbf{a}, \mathbf{b}, \mathbf{c}, a, b \in A$ the following identities are satisfied*:

(s1) $f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}, b, \mathbf{c}) = f^{\mathcal{A}}(\mathbf{a}, b, \mathbf{b}, a, \mathbf{c})$,
(s2) $f^{\mathcal{A}}(a, a, b, \mathbf{a}) = f^{\mathcal{A}}(a, b, b, \mathbf{a})$,
(s3) $f^{\mathcal{A}}(g^{\mathcal{A}}(a, \mathbf{a}), b, \mathbf{b}) = f^{\mathcal{A}}(g^{\mathcal{A}}(b, \mathbf{a}), a, \mathbf{b})$,
(s4) $f^{\mathcal{A}}(f^{\mathcal{A}}(a, \ldots, a), \mathbf{a}) = f^{\mathcal{A}}(a, \mathbf{a})$,
(s5) $f^{\mathcal{A}}(g^{\mathcal{A}}(a, b, \mathbf{a}), a, \mathbf{b}) = f^{\mathcal{A}}(g^{\mathcal{A}}(a, b, \mathbf{a}), b, \mathbf{b})$,
(s6) $f^{\mathcal{A}}(f^{\mathcal{A}}(g^{\mathcal{A}}(a, b, \mathbf{a}), \mathbf{b}), \mathbf{c}) = f^{\mathcal{A}}(g^{\mathcal{A}}(g^{\mathcal{A}}(a, \mathbf{b}, \overline{\mathbf{b}}), b, \mathbf{a}), \mathbf{c})$,
where $f \in \Sigma_m$, $g \in \Sigma_n$, $m \leqslant n$ and sequence $\overline{\mathbf{b}}$ consists of $n - m$ times b.

Proof. We are going to prove here only identities (s1) and (s5), the proofs for the other identities can be found in [11]. For (s1) we note that

$$\begin{aligned} f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}, b, \mathbf{c}) &= f^{\mathcal{A}}(\mathbf{a}, f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}, b, \mathbf{c}), \mathbf{b}, b, \mathbf{c}) \\ &= f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}, f^{\mathcal{A}}(\mathbf{a}, b, \mathbf{b}, b, \mathbf{c}), \mathbf{c}) \\ &= f^{\mathcal{A}}(\mathbf{a}, b, \mathbf{b}, f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}, b, \mathbf{c}), \mathbf{c}) \\ &= f^{\mathcal{A}}(\mathbf{a}, f^{\mathcal{A}}(\mathbf{a}, b, \mathbf{b}, a, \mathbf{c}), \mathbf{b}, b, \mathbf{c}) = p(f^{\mathcal{A}}(\mathbf{a}, b, \mathbf{b}, a, \mathbf{c})), \end{aligned}$$

where $p = f^{\mathcal{A}}(\mathbf{a}, \xi, \mathbf{b}, b, \mathbf{c})$. By the same argument and swapping a and b, it can be proved that $f^{\mathcal{A}}(\mathbf{a}, b, \mathbf{b}, a, \mathbf{c}) = q(f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}, b, \mathbf{c}))$ for some $q \in \mathrm{Tr}(\mathcal{A})$. Thus, from Lemma 2.15, it follows that $f^{\mathcal{A}}(\mathbf{a}, a, \mathbf{b}, b, \mathbf{c}) = f^{\mathcal{A}}(\mathbf{a}, b, \mathbf{b}, a, \mathbf{c})$.

Now, suppose (s2)–(s4) have been already proved [11].

For (s5) we distinguish two cases. First, suppose sequence **a** is empty. By using identities (s4), (s3), (s1), (s3), (s3) and (s4) consecutively, we get

$$\begin{aligned} f^{\mathcal{A}}(g^{\mathcal{A}}(a, b), a, \mathbf{b}) &= f^{\mathcal{A}}(g^{\mathcal{A}}(a, g^{\mathcal{A}}(b, b)), a, \mathbf{b}) \\ &= f^{\mathcal{A}}(g^{\mathcal{A}}(b, g^{\mathcal{A}}(a, b)), a, \mathbf{b}) = f^{\mathcal{A}}(g^{\mathcal{A}}(g^{\mathcal{A}}(a, b), b), a, \mathbf{b}) \\ &= f^{\mathcal{A}}(g^{\mathcal{A}}(g^{\mathcal{A}}(a, b), a), b, \mathbf{b}) = f^{\mathcal{A}}(g^{\mathcal{A}}(g^{\mathcal{A}}(a, a), b), b, \mathbf{b}) \\ &= f^{\mathcal{A}}(g^{\mathcal{A}}(a, b), b, \mathbf{b}). \end{aligned}$$

Second, suppose that sequence **a** is not empty and that it has the form $\mathbf{a} = (c, \mathbf{c})$. By using identities (s3), (s1), (s2) and (s3) consecutively, we get

$$\begin{aligned} f^{\mathcal{A}}(g^{\mathcal{A}}(a, b, \mathbf{a}), a, \mathbf{b}) &= f^{\mathcal{A}}(g^{\mathcal{A}}(a, b, c, \mathbf{c}), a, \mathbf{b}) \\ &= f^{\mathcal{A}}(g^{\mathcal{A}}(a, b, a, \mathbf{c}), c, \mathbf{b}) = f^{\mathcal{A}}(g^{\mathcal{A}}(a, a, b, \mathbf{c}), c, \mathbf{b}) \\ &= f^{\mathcal{A}}(g^{\mathcal{A}}(a, b, b, \mathbf{c}), c, \mathbf{b}) = f^{\mathcal{A}}(g^{\mathcal{A}}(a, b, c, \mathbf{c}), b, \mathbf{b}) \\ &= f^{\mathcal{A}}(g^{\mathcal{A}}(a, b, \mathbf{a}), b, \mathbf{b}). \qquad \square \end{aligned}$$

Definition 2.17. An ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$ is *symbolic* if it is a semilattice algebra and $f^{\mathcal{A}}(a_1, \ldots, a_m) \leqslant a_j$ holds for every $a_1, \ldots, a_m \in A$, $f \in \Sigma_m$ $(m > 0)$ and $j \leqslant m$.

The class of all semilattice Σ-algebras is denoted by $\mathbf{SL}(\Sigma)$ and $\mathbf{Sym}(\Sigma)$ denotes the class of all symbolic ordered Σ-algebras.

Proposition 2.18. *Class* $\mathbf{SL}(\Sigma)$ *is a variety of finite algebras and class* $\mathbf{Sym}(\Sigma)$ *is a variety of finite ordered algebras.*

3. Positive variety theorem

Recall that a ranked alphabet is a finite set of function symbols, and if Σ is a ranked alphabet, the set of m-ary function symbols from Σ is denoted by Σ_m (for every $m \geqslant 0$); in particular, Σ_0 is the set of constant symbols from Σ. For a ranked alphabet Σ and a *leaf*

alphabet X, the set $\mathrm{T}(\Sigma, X)$ of *ΣX-trees* is the smallest set satisfying
(1) $\Sigma_0 \cup X \subseteq \mathrm{T}(\Sigma, X)$, and
(2) $f(t_1, \ldots, t_m) \in \mathrm{T}(\Sigma, X)$ for all $m > 0$, $f \in \Sigma_m$, $t_1, \ldots, t_m \in \mathrm{T}(\Sigma, X)$.
Any subset of $\mathrm{T}(\Sigma, X)$ is a *tree language*.

The *ΣX-term algebra* $\mathcal{T}(\Sigma, X) = (\mathrm{T}(\Sigma, X), \Sigma)$ is defined by setting
(1) $c^{\mathcal{T}(\Sigma,X)} = c$ for each $c \in \Sigma_0$, and
(2) $f^{\mathcal{T}(\Sigma,X)}(t_1, \ldots, t_m) = f(t_1, \ldots, t_m)$ for all $m > 0$, function symbols $f \in \Sigma_m$ and trees $t_1, \ldots, t_m \in \mathrm{T}(\Sigma, X)$.

Let ξ be a (special) symbol which does not appear in any ranked alphabet or leaf alphabet considered here. The set of *ΣX-contexts*, denoted by $\mathrm{C}(\Sigma, X)$, consists of the $\Sigma(X \cup \{\xi\})$-trees in which ξ appears exactly once. For $P, Q \in \mathrm{C}(\Sigma, X)$ and $t \in \mathrm{T}(\Sigma, X)$ the context $Q \cdot P$, the composition of P and Q, results from P by replacing the special leaf ξ with Q, while the term $P(t)$, also denoted by $t \cdot P$, results from P by replacing ξ with t. Note that $\mathrm{C}(\Sigma, X)$ is a monoid with the composition operation, and that $t \cdot (Q \cdot P) = (t \cdot Q) \cdot P$ holds for all $P, Q \in \mathrm{C}(\Sigma, X)$, $t \in \mathrm{T}(\Sigma, X)$. There is a bijective correspondence between $\mathrm{C}(\Sigma, X)$ and translations of the term algebra $\mathrm{Tr}(\mathcal{T}(\Sigma, X))$ in a natural way: an elementary context $P = f(t_1, \ldots, \xi, \ldots, t_m)$ corresponds to $P^{\mathcal{T}(\Sigma,X)} = f^{\mathcal{T}(\Sigma,X)}(t_1, \ldots, \xi, \ldots, t_m)$, and the composition $P \cdot Q$ of the contexts P and Q corresponds to the composition $P^{\mathcal{T}(\Sigma,X)} \cdot Q^{\mathcal{T}(\Sigma,X)}$ of translations.

Definition 3.1. For a tree language $T \subseteq \mathrm{T}(\Sigma, X)$, the *syntactic quasi-order* $\preccurlyeq_T$ of T is defined by the following: for $t, s \in \mathrm{T}(\Sigma, X)$

$$t \preccurlyeq_T s \iff (\forall P \in \mathrm{C}(\Sigma, X))\big(s \cdot P \in T \;\Rightarrow\; t \cdot P \in T\big).$$

The corresponding equivalence relation $\theta_T = \preccurlyeq_T \cap \preccurlyeq_T^{-1}$ of $\preccurlyeq_T$ is the *syntactic congruence* of T

$$t \,\theta_T\, s \iff (\forall P \in \mathrm{C}(\Sigma, X))\big(t \cdot P \in T \;\Leftrightarrow\; s \cdot P \in T\big).$$

The *syntactic ordered algebra* of T is $\mathrm{SOA}(T) = (\mathrm{T}(\Sigma, X)/\theta_T, \Sigma, \leqslant_T)$, where $\leqslant_T$ is the order induced by $\preccurlyeq_T$: $t/\theta_T \leqslant_T s/\theta_T \Leftrightarrow t \preccurlyeq_T s$ for $t, s \in \mathrm{T}(\Sigma, X)$.

It can be easily seen that not every ordered algebra is the syntactic ordered algebra of a tree language. However, syntactic ordered algebras can be characterized as follows (cf. [22, Proposition 3.6]).

Proposition 3.2. *A finite ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$ is order isomorphic to the syntactic ordered algebra of a tree language if and only if there exists an ideal $I \trianglelefteq \mathcal{A}$ such that $\preccurlyeq_I = \leqslant$.*

Proof. First, suppose $\mathcal{A} \cong \mathrm{SOA}(T)$ for some tree language T. Then the subset $I = T/\theta_T = \{t/\theta_T \mid t \in T\}$ is an ideal of $\mathrm{SOA}(T)$ and $\preccurlyeq_I = \leqslant_T$ holds.

Conversely, suppose $\preccurlyeq_I = \leqslant$ for some $I \trianglelefteq \mathcal{A}$. Let the Σ-morphism $\varphi : \mathcal{T}(\Sigma, A) \to \mathcal{A}$ be obtained by extending the identity mapping $1_A : A \to A$. Since φ is an epimorphism, then $\preccurlyeq_{I\varphi^{-1}} = \varphi \circ \preccurlyeq_I \circ \varphi^{-1}$ by Proposition 2.8(3). Hence, Proposition 2.3 implies that $\mathcal{T}(\Sigma, A)/\preccurlyeq_{I\varphi^{-1}} \cong \mathcal{A}/\preccurlyeq_I$, and since $\preccurlyeq_I = \leqslant$, then $\mathrm{SOA}(I\varphi^{-1}) \cong \mathcal{A}$. $\square$

3.1. Recognizability by ordered algebras

Let Σ be a ranked alphabet, X be a leaf alphabet, and $\mathcal{A} = (A, \Sigma, \leqslant)$ be an ordered algebra. A tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is *recognized* by $\mathcal{A}$ if there exists an ideal $I \trianglelefteq \mathcal{A}$ and a Σ-morphism $\varphi : \mathcal{T}(\Sigma, X) \to \mathcal{A}$ such that $T = I\varphi^{-1}$. An initial assignment for $\mathcal{A}$ is a mapping $\alpha : X \to A$. It can be uniquely extended to an order morphism $\mathcal{T}(\Sigma, X) \to \mathcal{A}$ which is denoted by $\alpha^{\mathcal{A}}$. For an ideal $I \trianglelefteq \mathcal{A}$, the tree language recognized by $(\mathcal{A}, \alpha, I)$ is $\{t \in \mathrm{T}(\Sigma, X) \mid t\alpha^{\mathcal{A}} \in I\} = I(\alpha^{\mathcal{A}})^{-1}$.

Proposition 3.3. *For a tree language $T \subseteq \mathrm{T}(\Sigma, X)$ and an ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$, $\mathrm{SOA}(T) \prec \mathcal{A}$ if and only if T is recognized by $\mathcal{A}$.*

Proof. Suppose $T = I\varphi^{-1}$ for a morphism $\varphi : \mathcal{T}(\Sigma, X) \to \mathcal{A}$ and an ideal $I \trianglelefteq \mathcal{A}$. Let the ordered Σ-algebra $\mathcal{B}$ be the image of φ, and define the mapping $\psi : \mathcal{B} \to \mathrm{SOA}(T)$ by $(t\varphi)\psi = t/\theta_T$ for $t \in \mathrm{T}(\Sigma, X)$. We show that $t\varphi \leqslant s\varphi$ implies $t \preccurlyeq_T s$ for any $t, s \in \mathrm{T}(\Sigma, X)$. This also proves that ψ is well-defined. Suppose $t\varphi \leqslant s\varphi$, then $t\varphi \preccurlyeq_I s\varphi$ since $\leqslant \ \subseteq \ \preccurlyeq_I$. Now, for any $p \in \mathrm{Tr}(\mathcal{A})$,

$$\begin{aligned} p(s) \in T \Rightarrow p(s)\varphi \in I \quad &\Rightarrow \quad p_\varphi(s\varphi) \in I \quad \Rightarrow \quad p_\varphi(t\varphi) \in I \\ \Rightarrow p(t)\varphi \in I \quad &\Rightarrow \quad p(t) \in T, \end{aligned}$$

so $t \preccurlyeq_T s$. It can also be seen that ψ is a Σ-morphism. Thus ψ is an order epimorphism, hence $\mathrm{SOA}(T) \leftarrow \mathcal{B} \subseteq \mathcal{A}$.

Now suppose for an ordered algebra $\mathcal{B}$ that $\mathrm{SOA}(T) \leftarrow \mathcal{B} \subseteq \mathcal{A}$, and let $\psi : \mathcal{B} \to \mathrm{SOA}(T)$ be an order epimorphism. A Σ-morphism $\varphi : \mathcal{T}(\Sigma, X) \to \mathcal{A}$ can be defined by choosing $x\varphi \in B$, such that $(x\varphi)\psi = x/\theta_T$ for every $x \in X \cup \Sigma_0$. By induction on t it can be shown that $t\varphi\psi = t/\theta_T$ holds for every $t \in \mathrm{T}(\Sigma, X)$. The set $\{t/\theta_T \in \mathrm{SOA}(T) \mid t \in T\}\psi^{-1}$ is an ideal of $\mathcal{B}$. If I is the ideal of $\mathcal{A}$ generated by this set, then $T = I\varphi^{-1}$. $\square$

From Proposition 3.3 it follows that the syntactic ordered algebra of a tree language is the least ordered algebra which recognizes the tree language.

Let us recall that for a tree language $T \subseteq \mathrm{T}(\Sigma, X)$, a context $P \in \mathrm{C}(\Sigma, X)$, and a Σ-morphism $\varphi : \mathcal{T}(\Sigma, Y) \to \mathcal{T}(\Sigma, X)$, the inverse translation of T under P is $P^{-1}(T) = \{t \in \mathrm{T}(\Sigma, X) \mid t \cdot P \in T\}$, and the inverse morphism of T under φ is $T\varphi^{-1} = \{t \in \mathrm{T}(\Sigma, Y) \mid t\varphi \in T\}$ (cf. [22]).

The following is an immediate consequence of Corollary 2.9.

Corollary 3.4. *For tree languages $T, T' \subseteq \mathrm{T}(\Sigma, X)$, a context $P \in \mathrm{C}(\Sigma, X)$, and a Σ-morphism $\varphi : \mathcal{T}(\Sigma, Y) \to \mathcal{T}(\Sigma, X)$,*
(1) $\mathrm{SOA}(T \cap T'), \mathrm{SOA}(T \cup T') \prec \mathrm{SOA}(T) \times \mathrm{SOA}(T')$;
(2) $\mathrm{SOA}(P^{-1}(T)) \leftarrow \mathrm{SOA}(T)$;
(3) $\mathrm{SOA}(T\varphi^{-1}) \prec \mathrm{SOA}(T)$, *and if φ is surjective then* $\mathrm{SOA}(T\varphi^{-1}) \cong \mathrm{SOA}(T)$.

3.2. Positive variety theorem

Let Σ be a fixed ranked alphabet. Let us recall that a class of finite ordered Σ-algebras is a variety (of finite ordered algebras) if it is closed under order subalgebras, order epimorphic images, and finite direct products.

Definition 3.5. An *indexed family* of recognizable tree languages is a family $\mathscr{V} = \{\mathscr{V}(X)\}$ where $\mathscr{V}(X)$ consists of recognizable ΣX-tree languages for any leaf alphabet X. An indexed family is a *positive variety of tree languages*, abbreviated by PVTL, if it is closed under finite positive Boolean operations (finite intersections and unions), inverse translations, and inverse morphisms.

Definition 3.6. For a variety of finite ordered algebras $\mathscr{K}$, let the indexed family $\mathscr{K}^{\mathtt{t}} = \{\mathscr{K}^{\mathtt{t}}(X)\}$ be the family of tree languages whose syntactic ordered algebras are in $\mathscr{K}$, that is

$$\mathscr{K}^{\mathtt{t}}(X) = \{T \subseteq \mathrm{T}(\Sigma, X) \mid \mathrm{SOA}(T) \in \mathscr{K}\}.$$

For a positive variety of tree languages $\mathscr{V}$, let $\mathscr{V}^{\mathtt{a}}$ be the variety of finite ordered algebras generated by syntactic ordered algebras of tree languages in $\mathscr{V}$, that is $\mathscr{V}^{\mathtt{a}}$ is the VFOA generated by the class

$$\{\mathrm{SOA}(T) \mid T \in \mathscr{V}(X) \text{ for a leaf alphabet } X\}.$$

By Corollary 3.4, for a variety of finite ordered algebras $\mathscr{K}$, the family $\mathscr{K}^{\mathtt{t}}$ is a positive variety of tree languages.

Lemma 3.7. *Let $\mathscr{K}$ and $\mathscr{L}$ be PVTLs, and let $\mathscr{V}$ and $\mathscr{W}$ be VFOAs.*
(1) *The operations $\mathscr{K} \mapsto \mathscr{K}^{\mathtt{t}}$ and $\mathscr{V} \mapsto \mathscr{V}^{\mathtt{a}}$ are monotone, i.e., if $\mathscr{K} \subseteq \mathscr{L}$ and $\mathscr{V} \subseteq \mathscr{W}$, then $\mathscr{K}^{\mathtt{t}} \subseteq \mathscr{L}^{\mathtt{t}}$ and $\mathscr{V}^{\mathtt{a}} \subseteq \mathscr{W}^{\mathtt{a}}$.*
(2) *$\mathscr{V} \subseteq \mathscr{V}^{\mathtt{at}}$ and $\mathscr{K}^{\mathtt{ta}} \subseteq \mathscr{K}$.*

Proof. The statement (1) and the inclusion $\mathscr{V} \subseteq \mathscr{V}^{\mathtt{at}}$ are obvious. In order to prove $\mathscr{K}^{\mathtt{ta}} \subseteq \mathscr{K}$, we note that if $\mathcal{A} \in \mathscr{K}^{\mathtt{ta}}$ then $\mathcal{A} \prec \mathrm{SOA}(T_1) \times \cdots \times \mathrm{SOA}(T_n)$ for some $T_1, \ldots, T_n$ in $\mathscr{K}^{\mathtt{t}}$, what by definition means that $\mathrm{SOA}(T_j) \in \mathscr{K}$ for every j, and hence $\mathcal{A} \in \mathscr{K}$. $\square$

The following was proved for classical algebras in [18].

Lemma 3.8. *For any finite ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$ there are tree languages $T_1, \ldots, T_m$ recognizable by $\mathcal{A}$, such that*

$$\mathcal{A} \subseteq \mathrm{SOA}(T_1) \times \cdots \times \mathrm{SOA}(T_m).$$

Proof. Let $\mathcal{A} = (A, \Sigma, \leqslant)$ be a finite ordered algebra, and suppose the epimorphism $\psi : \mathcal{T}(\Sigma, A) \to \mathcal{A}$ is obtained by extending the identity mapping $1_A : A \to A$. Recall that for any $a \in A$, $(a] = \{b \in A \mid b \leqslant a\}$ is the ideal of $\mathcal{A}$ generated by a. By Corollary 2.9(3), $\mathrm{SOA}((a]\psi^{-1}) \cong \mathcal{A}/(a]$ for every $a \in A$. We are proving $\mathcal{A} \subseteq \prod_{a \in A} \mathcal{A}/(a]$. This will finish the proof since $(a]\psi^{-1}$ is recognizable by $\mathcal{A}$. Define the mapping $\phi : \mathcal{A} \to \prod_{a \in A} \mathcal{A}/(a]$ by $u\phi = \left(u/\theta_{(a]}\right)_{a \in A}$ for $u \in A$. Clearly ϕ is an order morphism. It suffices to show that ϕ is injective. Suppose $u\phi = v\phi$ for $u, v \in A$. Then $u/\theta_{(a]} = v/\theta_{(a]}$ for every $a \in A$. In particular $u/\theta_{(u]} = v/\theta_{(u]}$ and $u/\theta_{(v]} = v/\theta_{(v]}$, what imply $v \in (u]$ and $u \in (v]$, respectively. So, $u \leqslant v$ and $v \leqslant u$, thus $u = v$. $\square$

Corollary 3.9. (1) *Every VFOA is generated by syntactic ordered algebras of some tree languages.*

(2) *For any PVTL $\mathscr{V}$ and any finite ordered algebra $\mathcal{A}$, if every tree language recognizable by $\mathcal{A}$ belongs to $\mathscr{V}$, then $\mathcal{A} \in \mathscr{V}^{\mathtt{a}}$.*

Lemma 3.10. *For every variety of finite ordered algebras $\mathscr{K}$, $\mathscr{K} \subseteq \mathscr{K}^{\mathtt{ta}}$.*

Proof. By Corollary 3.9(1), it is enough to show that syntactic ordered algebras of tree languages that belong to $\mathscr{K}$ are in $\mathscr{K}^{\mathtt{ta}}$. Suppose $\mathrm{SOA}(T) \in \mathscr{K}$ for a tree language T. Then T is in $\mathscr{K}^{\mathtt{t}}$ by definition, so $\mathrm{SOA}(T) \in \mathscr{K}^{\mathtt{ta}}$, which finishes the proof. $\square$

The essential part of the positive variety theorem is the following.

Lemma 3.11. *For every positive variety of tree languages $\mathscr{V}$, $\mathscr{V}^{\mathtt{at}} \subseteq \mathscr{V}$.*

Proof. Suppose $T \in \mathscr{V}^{\mathtt{at}}(X)$. Then there are leaf alphabets $X_1, \ldots, X_n$ and tree languages $T_1 \in \mathscr{V}(X_1), \ldots, T_n \in \mathscr{V}(X_n)$, such that $\mathrm{SOA}(T)$ divides the product $\mathcal{A} = \mathrm{SOA}(T_1) \times \cdots \times \mathrm{SOA}(T_n)$. Thus, by Proposition 3.3, T is recognized by $\mathcal{A}$, and so there is an order morphism $\varphi : \mathcal{T}(\Sigma, X) \to \mathcal{A}$ and an ideal $I \trianglelefteq \mathcal{A}$ such that $T = I\varphi^{-1}$. Let $\mathrm{SOA}(T_j) = \mathcal{A}_j = (A_j, \Sigma, \leqslant_j)$ for each $j \leqslant n$.

For any $\mathbf{a} = (a_1, \ldots, a_n) \in \prod_{i=1}^n A_i$ we have $(\mathbf{a}] = (a_1] \times \cdots \times (a_n]$. Let $\varphi_j : \mathcal{T}(\Sigma, X) \to \mathcal{A}_j$ be the composition of φ with the jth projection mapping $\prod_{i=1}^n A_i \to A_j$. Then $T = I\varphi^{-1} = \bigcup_{\mathbf{a} \in I} (\mathbf{a}]\varphi^{-1} = \bigcup_{(a_1,\ldots,a_n) \in I} \bigcap_{j \leqslant n} (a_j]\varphi_j^{-1}$.

We aim at showing $T \in \mathscr{V}(X)$. It is enough to show $(a_j]\varphi_j^{-1} \in \mathscr{V}(X)$ for every $j \leqslant n$. Fix a $j \leqslant n$. Let $\varphi_{T_j} : \mathcal{T}(\Sigma, X_j) \to \mathcal{A}_j$ be the syntactic morphism of T_j. A Σ-morphism $\chi_j : \mathcal{T}(\Sigma, X) \to \mathcal{T}(\Sigma, X_j)$, such that $\chi_j \varphi_{T_j} = \varphi_j$ can be constructed. Then $(a_j]\varphi_j^{-1} = (a_j]\varphi_{T_j}^{-1}\chi_j^{-1}$ and, since $\mathscr{V}$ is closed under inverse morphisms, for showing $(a_j]\varphi_j^{-1} \in \mathscr{V}(X)$ it suffices to show $(a_j]\varphi_{T_j}^{-1} \in \mathscr{V}(X_j)$. Choose a $t \in \mathrm{T}(\Sigma, X_j)$, such that $a_j = t\varphi_{T_j}$. We show

$$(a_j]\varphi_{T_j}^{-1} = \bigcap\{P^{-1}(T_j) \mid P \in \mathrm{C}(\Sigma, X_j),\, P(t) \in T_j\}.$$

The intersection on the right-hand side is finite since T_j is recognizable. For any $s \in \mathrm{T}(\Sigma, X_j)$, we have that $s \in (a_j]\varphi_{T_j}^{-1}$ iff $s\varphi_{T_j} \leqslant_j a_j = t\varphi_{T_j}$, i.e., $s \preccurlyeq_{T_j} t$, what by definition means that $P(t) \in T_j$ implies $P(s) \in T_j$ for any $P \in \mathrm{C}(\Sigma, X_j)$. This is further equivalent to $s \in P^{-1}(T_j)$ whenever $P(t) \in T_j$ for any $P \in \mathrm{C}(\Sigma, X_j)$, what finally means $s \in \bigcap\{P^{-1}(T_j) \mid P \in \mathrm{C}(\Sigma, X_j), P(t) \in T_j\}$. From $T_j \in \mathscr{V}(X_j)$ and the fact that $\mathscr{V}$ is closed under inverse translations and positive Boolean operations, it follows that $(a_j]\varphi_{T_j}^{-1} \in \mathscr{V}(X_j)$. Therefore, $(a_j]\varphi_j^{-1}$ belongs to $\mathscr{V}(X)$ for any j, thus $T \in \mathscr{V}(X)$. $\square$

Summing up, we have shown the following.

Proposition 3.12 (*Positive Variety Theorem*). *The operations $\mathscr{K} \mapsto \mathscr{K}^{\mathtt{t}}$ and $\mathscr{V} \mapsto \mathscr{V}^{\mathtt{a}}$ are mutually inverse lattice isomorphisms between the class of all varieties of finite ordered* *algebras and the class of all positive varieties of recognizable tree languages, i.e.,* $\mathscr{V}^{\mathtt{at}} = \mathscr{V}$ *and* $\mathscr{K}^{\mathtt{ta}} = \mathscr{K}$.

3.3. Examples

Families of tree languages that correspond, in the sense of Positive Variety Theorem (Proposition 3.12), to varieties of algebras introduced earlier are studied here.

3.3.1. Cofinite tree languages

Definition 3.13. A tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is *cofinite* if it is empty or its complement $\mathrm{T}(\Sigma, X) \setminus T$ is finite. The family of cofinite ΣX-tree languages is denoted by $\mathrm{Cof}(\Sigma, X)$, and $\mathrm{Cof}_\Sigma = \{\mathrm{Cof}(\Sigma, X)\}$ is the family of cofinite tree languages for all leaf alphabets X.

Proposition 3.14. *A language* $T \subseteq \mathrm{T}(\Sigma, X)$ *is cofinite if and only if it can be recognized by a finite ordered nilpotent algebra.*

Proof. Suppose $T \subseteq \mathrm{T}(\Sigma, X)$ is cofinite. There exists an $n \in \mathbb{N}$ such that $P_1 \cdots P_n(t) \in T$ holds for all $P_1, \ldots, P_n \in \mathrm{C}(\Sigma, X) \setminus \{\xi\}$ and $t \in \mathrm{T}(\Sigma, X)$. Therefore, $P_1 \cdots P_n(t) \preccurlyeq_T s$ holds for all $P_1, \ldots, P_n \in \mathrm{C}(\Sigma, X) \setminus \{\xi\}$ and $t, s \in \mathrm{T}(\Sigma, X)$. This immediately implies that the syntactic algebra $\mathrm{SOA}(T)$ satisfies $p_1 \cdots p_n(a) \leqslant_T b$ for all $p_1, \ldots, p_n \in \mathrm{TrS}(\mathrm{SOA}(T))$ and $a, b \in \mathrm{SOA}(T)$. Thus, $\mathrm{SOA}(T)$ is an ordered n-nilpotent algebra.

Conversely, suppose that a tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is recognized by an ordered n-nilpotent algebra $\mathcal{A} = (A, \Sigma, \leqslant)$. Let $\varphi : \mathrm{T}(\Sigma, X) \to \mathcal{A}$ be an order morphism and $I \trianglelefteq A$ be an ideal, such that $T = I\varphi^{-1}$. The mapping $\varphi_* : \mathrm{C}(\Sigma, X) \setminus \{\xi\} \to \mathrm{TrS}(\mathcal{A})$ obtained from setting $f(t_1, \ldots, \xi, \ldots, t_m)\varphi_* = f^{\mathcal{A}}(t_1\varphi, \ldots, \xi, \ldots, t_m\varphi)$ for all $f \in \Sigma_m$ ($m > 0$) and $t_1, \ldots, t_m \in \mathrm{T}(\Sigma, X)$, and $(P \cdot Q)\varphi_* = P\varphi_* \cdot Q\varphi_*$, is a semigroup morphism which satisfies $P\varphi_*(t\varphi) = P(t)\varphi$ for all $t \in \mathrm{T}(\Sigma, X)$, $P \in \mathrm{C}(\Sigma, X) \setminus \{\xi\}$. Since $\mathcal{A}$ is an ordered n-nilpotent algebra, then $p_1 \cdots p_n(a) \in I$ holds for all $p_1, \ldots, p_n \in \mathrm{TrS}(\mathcal{A})$ and $a \in A$. In particular, $P_1\varphi_* \cdots P_n\varphi_*(t\varphi) \in I$ holds for all $P_1, \ldots, P_n \in \mathrm{C}(\Sigma, X) \setminus \{\xi\}$ and $t \in \mathrm{T}(\Sigma, X)$, i.e., $P_1 \cdots P_n(t)\varphi \in I$, and so $P_1 \cdots P_n(t) \in I\varphi^{-1} = T$. Hence, T is cofinite. $\square$

Corollary 3.15. *Family* Cof_Σ *is a PVTL and* $\mathrm{Cof}_\Sigma = \mathbf{Nil}(\Sigma)^{\mathtt{t}}$.

Proof. This follows immediately from Propositions 3.14, 2.12 and 3.12. $\square$

3.3.2. Semilattice and symbolic tree languages

We can assume that the leaf alphabets X are always disjoint from the ranked alphabet Σ.

Definition 3.16. For a tree $t \in \mathrm{T}(\Sigma, X)$, the *contents* $\mathrm{c}(t)$ of t is the set of symbols from $\Sigma \cup X$ which appear in t. It can be defined inductively as:
(1) $\mathrm{c}(x) = \{x\}$ for $x \in \Sigma_0 \cup X$;
(2) $\mathrm{c}(f(t_1, \ldots, t_m)) = \{f\} \cup \mathrm{c}(t_1) \cup \cdots \cup \mathrm{c}(t_m)$ for $t_1, \ldots, t_m \in \mathrm{T}(\Sigma, X)$ and $f \in \Sigma_m$.
For a subset $Z \subseteq \Sigma \cup X$, the tree language $T(Z)$ consists of trees in which all symbols from Z appear, i.e.,

$$T(Z) = \{t \in \mathrm{T}(\Sigma, X) \mid Z \subseteq \mathrm{c}(t)\}.$$

A tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is *symbolic* if it is a finite union of tree languages of the form $T(Z)$ for some subsets $Z \subseteq \Sigma \cup X$. The family of all symbolic ΣX-tree languages is denoted by $\mathrm{Sym}(\Sigma, X)$, and $\mathrm{Sym}_\Sigma = \{\mathrm{Sym}(\Sigma, X)\}$ is the family of symbolic tree languages for all leaf alphabets X.

Lemma 3.17. *For a tree language $T \subseteq \mathrm{T}(\Sigma, X)$ the following properties are equivalent*:
(1) *T is symbolic*;
(2) *for all trees $t, t' \in \mathrm{T}(\Sigma, X)$, $\mathrm{c}(t) \subseteq \mathrm{c}(t')$ and $t \in T$ imply $t' \in T$*;
(3) $T = \bigcup_{t \in T} T(\mathrm{c}(t))$.

Proof. The implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are straightforward. For the implication (2) $\Rightarrow$ (3), the inclusion $T \subseteq \bigcup_{t \in T} T(\mathrm{c}(t))$ always holds. Suppose $t' \in T(\mathrm{c}(t))$ for some $t \in T$. Then $\mathrm{c}(t) \subseteq \mathrm{c}(t')$, and so $t' \in T$, what implies $\bigcup_{t \in T} T(\mathrm{c}(t)) \subseteq T$. $\square$

Definition 3.18. A tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is a *semilattice tree language* if $\mathrm{c}(t) = \mathrm{c}(t')$ and $t \in T$ imply $t' \in T$ for all $t, t' \in \mathrm{T}(\Sigma, X)$. The family of semilattice ΣX-tree languages is denoted by $\mathrm{SL}(\Sigma, X)$, and $\mathrm{SL}_\Sigma = \{\mathrm{SL}(\Sigma, X)\}$ is the family of semilattice tree languages for all leaf alphabets X.

The rest of this subsection is devoted to proving the fact that semilattice and symbolic tree languages are definable by semilattice and symbolic algebras respectively, i.e., $\mathrm{SL}_\Sigma = \mathbf{SL}(\Sigma)^{\mathtt{t}}$ and $\mathrm{Sym}_\Sigma = \mathbf{Sym}(\Sigma)^{\mathtt{t}}$.

Fix a ranked alphabet Σ and a leaf alphabet X. Finite sequences of trees are denoted by bold face letters, e.g., $\mathbf{t}$ is a sequence $\langle t_1, \ldots, t_m \rangle$ for some trees $t_1, \ldots, t_m \in \mathrm{T}(\Sigma, X)$.

Let σ be a Σ-congruence on $\mathcal{T}(\Sigma, X)$ such that $\mathcal{T}(\Sigma, X)/\sigma$ is a semilattice algebra, i.e., it satisfies the following relations for all function symbols $f, g \in \Sigma$ and trees $\mathbf{t}, \mathbf{r}, \mathbf{u}, \mathbf{v}$, $t \in \mathrm{T}(\Sigma, X)$:
(d1) $f(\mathbf{t}, f(\mathbf{t}, t, \mathbf{r}), \mathbf{r}) \ \sigma \ f(\mathbf{t}, t, \mathbf{r})$,
(d2) $f(\mathbf{t}, g(\mathbf{u}, t, \mathbf{v}), \mathbf{r}) \ \sigma \ g(\mathbf{u}, f(\mathbf{t}, t, \mathbf{r}), \mathbf{v})$.
In particular, as a corollary of Lemma 2.16, algebra $\mathcal{T}(\Sigma, X)/\sigma$ satisfies identities (s1)–(s6) of Lemma 2.16.

The family of Σ-congruences on $\mathcal{T}(\Sigma, X)$ satisfying (d1) and (d2) is closed under intersections and contains the universal relation $\mathrm{T}(\Sigma, X) \times \mathrm{T}(\Sigma, X)$, and so has the smallest element τ. Our aim is to prove that τ is determined by

$$t_1 \, \tau \, t_2 \iff \mathrm{c}(t_1) = \mathrm{c}(t_2)$$

for any trees t_1 and t_2.

Suppose that the elements of $\Sigma \setminus \Sigma_0$ are linearly ordered in such a way that function symbols with smaller arity are smaller than function symbols with greater arity. Assume also that the leaves $X \cup \Sigma_0$ are linearly ordered.

Let $\mathrm{c}_\Sigma(t) = (\Sigma \setminus \Sigma_0) \cap \mathrm{c}(t)$ be the set of nodes of a tree $t \in \mathrm{T}(\Sigma, X)$ and $\mathrm{c}_X(t) = (X \cup \Sigma_0) \cap \mathrm{c}(t)$ be its set of leaves.

A tree t is in the *canonical form* if

(1) either $t \in X \cup \Sigma_0$, or

(2) $t = f(t_1, x_2, \ldots, x_m)$ where

   (a) t_1 is in the canonical form and $x_2 \leqslant \cdots \leqslant x_m \in \Sigma_0 \cup X$,

   (b) f is the smallest in $\mathrm{c}_\Sigma(t)$,

   (c) either $f \notin \mathrm{c}_\Sigma(t_1)$ or $\mathrm{c}_\Sigma(t_1) = \{f\}$ and then $|\mathrm{c}_X(t_1)| > 1$,

   (d) if $|\mathrm{c}_X(t)| > m-1$ then $x_2 \lneqq \cdots \lneqq x_m$ are the smallest $m-1$ elements in $\mathrm{c}_X(t)$, and

   (e) otherwise if $\mathrm{c}_X(t) = \{x_2, \ldots, x_n\}$ with $n \leqslant m$, then $x_2 \lneqq \cdots \lneqq x_n$, $x_{n+1} = \cdots = x_m = x_n$ and $\mathrm{c}_X(t_1) = \{x_n\}$.

In other words, a tree is in the canonical form if on each its level only the leftmost node may be from $\Sigma \setminus \Sigma_0$, all the others are leaves from $\Sigma_0 \cup X$, nodes grow from the root downwards and leaves grow from left to right and from top to down. As soon as the set of nodes or leaves is exhausted, the last symbol from the exhausted set is repeated as long as there are still symbols in the other set to be used.

Let us fix σ to be any congruence on $\mathcal{T}(\Sigma, X)$ satisfying (d1) and (d2). Our aim is to show that every tree t is σ-equivalent to a tree t' in the canonical form, where $c(t) = c(t')$. A tree is called *leftmost branching* if its every subtree is either a leaf or of the form, $f(t, \mathbf{x})$, where t is a tree and $\mathbf{x}$ is a sequence of leaves (from $X \cup \Sigma_0$). For a tree t, the root of t, in notation $\mathrm{root}(t)$, is its topmost symbol. Transformation of a tree into a σ-equivalent tree in the canonical form consists of the following steps.

Step 1: *Shaping the tree into a leftmost branching tree while arranging the nodes in the increasing order from top to down*: We show that this can be done by induction on the number of nodes and leaves in t. The claim clearly holds for $t \in \Sigma_0 \cup X$. Suppose that $t = f(t_1, t_2, \ldots, t_n)$ where $t_1, \ldots, t_m$ have the shape of a leftmost branching tree and the nodes are in increasing order. Let $g = \min\{\mathrm{root}(t_1), \ldots, \mathrm{root}(t_m)\}$. Without loosing generality, by (s1), we can assume that $g = \mathrm{root}(t_1)$, and let $t_1 = g(t_1', x_2, \ldots, x_n)$. We distinguish two cases

If $g \leqslant f$ then by (d2),

$$t = f(g(t_1', x_2, \ldots, x_n), t_2, \ldots, t_m) \;\sigma\; g(f(t_1', t_2, \ldots, t_m), x_2, \ldots, x_n)$$

and now we can apply the induction hypotheses to $f(t_1', t_2, \ldots, t_m)$.

If $f < g$ then $m \leqslant n$ and by (s3), we have

$$\begin{aligned} t &= f(g(t_1', x_2, \ldots, x_n), t_2, \ldots, t_m) \\ &\quad \sigma f(g(t_1', t_2, \ldots, t_m, x_2, \ldots, x_{n-m+1}), x_{n-m+2}, \ldots, x_n) \end{aligned}$$

and then we can continue by induction.

We get a tree of the desired shape with nodes increasing from top to down, but there may be repetitions of same nodes.

Step 2: *Removing repetitions of nodes different from the greatest node*: The clause (s6) of Lemma 2.16 provides a transformation that pushes repetitions, i.e., if $f \leqslant g$ and ffg is a subsequence of the sequence of nodes, then the transformation will replace an extra copy of f by a copy of g. Namely, let $f_1, \ldots, f_{i-1}, f_i, \ldots, f_i, f_{i+1}, \ldots, f_k, k \in \mathbb{N}$, be the sequence of nodes read from the root downwards after Step 1, and assume that f_i is the first repeated symbol. By applying (s6) from Lemma 2.16, the last copy of f_i is replaced by a new copy of f_{i+1}. This is repeated as long as there is more than one f_i in the sequence. Thus, all repetitions of f_i are replaced by repetitions of f_{i+1}. After that, the last copy of f_{i+1} is replaced by a new copy of f_{i+2}, etc. Finally, only the last symbol f_k may have multiple copies, all the others appear only once.

After these transformations we get a tree σ-equivalent to t, branching only in the leftmost node and with increasing nodes where only the greatest node is possibly repeated. The tree is still not in the canonical form since leaves are not necessarily already arranged.

Step 3: *Arranging leaves into increasing order*: The sequence of leaves is read starting from left to right and from top downwards. This sequence can be sorted using standard algorithms for sorting sequences what assumes comparing the first symbol with the rest one by one and when a smaller one appears swap them and continue comparing the new first symbol with the rest of the sequence. After this the smallest leaf is on the first place. Repeat the same with the second one and the rest of the sequence, etc. We note that this swapping is supported by σ, since places of leaves on the same level can be changed by (s1), and if they are on different levels then (s3) is applied.

After this, leaves will be in increasing order, but there are possibly repetitions of those leaves which are not the greatest.

Step 4: *Removing repetitions of leaves different from the greatest leaf*: The idea is the same as in Step 2, the repetition of a smaller leaf is replaced by a repetition of the next greater leaf, so that repetitions are pushed trough the sequence and finally only the greatest leaf may be repeated. In other words, if $x < y$ then the subsequence of leaves of the form xxy is replaced by xyy. We distinguish four cases.

First, xxy appears on the same level, i.e., as the components of the same node. This case is solved by applying (s2).

Second, the first x is on one level and the second x and y are both on the next. This is solved easily by applying first (s1), then (s5) and so changing the first x into y, then applying (s3) to swap x and outer y, and finally once more (s1):

$$\begin{aligned}&f(g(t,x,y,\mathbf{x}),\mathbf{y},x)\ \sigma\ f(g(t,x,y,\mathbf{x}),x,\mathbf{y})\\ &\quad\sigma f(g(t,x,y,\mathbf{x}),y,\mathbf{y})\ \sigma\ f(g(t,y,y,\mathbf{x}),x,\mathbf{y})\ \sigma\ f(g(t,y,y,\mathbf{x}),\mathbf{y},x).\end{aligned}$$

Third, both x's are on the upper level and y is on the lower. We proceed as

$$\begin{aligned}&f(g(t,y,\mathbf{x}),\mathbf{y},x,x)\ \sigma\ f(g(x,y,\mathbf{x}),\mathbf{y},x,t)\\ &\quad\sigma f(g(x,y,\mathbf{x}),\mathbf{y},y,t)\ \sigma\ f(g(t,y,\mathbf{x}),\mathbf{y},y,x)\ \sigma\ f(g(t,y,\mathbf{x}),\mathbf{y},x,y).\end{aligned}$$

Note that t plays an important role here and existence of such a symbol follows from the fact that $f \leqslant g$ and thus the arity of g is at least 2.

Fourth, all three leaves appear on different levels. The tree is of the form $f(g(h(t,y,\mathbf{z}), x),x)$ where $f, g \in \Sigma_2$, and so the arity of h is at least two. The first x should be changed into y. The transformation is:

$$f(g(h(t,y,\mathbf{z}),x),x)\ \sigma\ f(g(h(x,y,\mathbf{z}),t),x)$$
$$\sigma f(g(h(x,y,\mathbf{z}),x),t)\ \sigma\ f(g(h(x,y,\mathbf{z}),y),t)\ \sigma\ f(g(h(x,y,\mathbf{z}),t),y)\ \sigma$$
$$\sigma f(g(h(t,y,\mathbf{z}),x),y)\ \sigma\ f(g(h(t,y,\mathbf{z}),y),x).$$

After this, our tree almost has the canonical form, the only disturbing thing may be too long subtree at the end having only the greatest symbol from $c_\Sigma(t)$ as nodes and the greatest element from $c_X(t)$ as leaves.

Step 5: *Fold the unnecessary part*: Applying (s4) as many times as needed the tree is folded into one without repetitions of the greatest symbol from $c_\Sigma(t)$, or with its repetitions but not with only the greatest element of $c_X(t)$ as leaves on the deepest level.
This finishes the procedure.

Clearly, the procedure results in a unique tree in the canonical form which is σ-equivalent to a given tree.

For example, suppose $h \in \Sigma_3$, $f, g \in \Sigma_2$, $c \in \Sigma_0$, $x \in X$, and the orders of symbols are $f < g < h$ and $x < c$. Let $t = h(g(x, f(x,c)), x, g(x,c))$. Then by applying the above steps we get the tree r_j in the jth step as follows:

$$\begin{aligned} t\ \sigma\ \ r_1 &= f(g(g(h(x,x,x),c),x),c)\\ \sigma\ \ r_2 &= f(g(h(h(x,c,x),x,x),x),c)\\ \sigma\ \ r_3 &= f(g(h(h(c,c,x),x,x),x),x)\\ \sigma\ \ r_4 &= f(g(h(h(c,c,c),c,c),c),x)\\ \sigma\ \ r_5 &= f(g(h(c,c,c),c),x). \end{aligned}$$

It can be noticed that the canonical form tree corresponding to a given tree t is determined by $\mathrm{c}(t)$ and can be constructed directly from this set. The procedure can roughly be described as follows:

1. put the smallest node in the root of the tree, draw the necessary branches, put the next smallest symbol from $c_\Sigma(t)$ in the left most node, continue doing this as long as $c_\Sigma(t)$ is not exhausted;
2. put the smallest leaf in the topmost leftmost free place, choose the next smallest and put in the next place, etc., as long as there are free places in the tree or the set $c_X(t)$ of leaves is not empty;
3. if not all $c_X(t)$ is used, continue building the tree by shifting all symbols on the last level by one place to the right, return the last leaf to $c_X(t)$, put the greatest element of $c_\Sigma(t)$ to the leftmost place, add its arity new branches, fill them with remaining symbols from $c_X(t)$ in the manner explained in 2, and repeat this step until the whole $c_X(t)$ is used;
4. if there are still free places put the greatest symbol from $c_X(t)$ there.

Recall that τ denotes the smallest congruence satisfying (d1) and (d2).

Lemma 3.19. *For any trees t_1 and t_2, $t_1 \tau t_2 \iff \mathrm{c}(t_1) = \mathrm{c}(t_2)$.*

Proof. Define τ' by $t_1\tau' t_2$ iff $\mathrm{c}(t_1) = \mathrm{c}(t_2)$. Obviously τ' satisfies (d1) and (d2). Let σ be any congruence satisfying (d1) and (d2). We are proving $\tau' \subseteq \sigma$. Assume $t_1\tau' t_2$. There are trees t_1' and t_2' in canonical form such that $t_1\sigma t_1'$ and $t_2\sigma t_2'$. Then $\mathrm{c}(t_1') = \mathrm{c}(t_1) = \mathrm{c}(t_2) = \mathrm{c}(t_2')$ and since the canonical tree is uniquely determined by its contents, it follows that $t_1' = t_2'$

which immediately implies that $t_1 \sigma t_2$. Therefore, τ' is the smallest congruence satisfying (d1) and (d2), and thus $\tau = \tau'$. □

For a context $P \in \mathrm{C}(\Sigma, X)$, the contents $\mathrm{c}(P)$ of P is the set of symbols from $\Sigma \cup X$ that appear in P. We note that $\mathrm{c}(P(t)) = \mathrm{c}(P) \cup \mathrm{c}(t)$ holds for any context $P \in \mathrm{C}(\Sigma, X)$ and tree $t \in \mathrm{T}(\Sigma, X)$.

Proposition 3.20. (1) *A tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is semilattice if and only if it is recognizable by a finite semilattice algebra.*

(2) *A tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is symbolic if and only if it is recognizable by a finite symbolic ordered algebra.*

Proof. (1) Since semilattice algebras form a variety of finite algebras, it suffices to prove that a tree language is semilattice iff its syntactic algebra is semilattice. By Lemma 3.19, T is a semilattice tree language iff $\tau \subseteq \theta_T$ iff the syntactic algebra of T is a semilattice algebra.

(2) Similarly to (1), it suffices to prove that a tree language is symbolic iff its syntactic ordered algebra is symbolic. Every symbolic tree language is also a semilattice tree language. So, if T is symbolic then the syntactic algebra of T is semilattice. On the other hand, since $\mathrm{c}(t) \subseteq \mathrm{c}(P(t))$ holds for all $t \in \mathrm{T}(\Sigma, X)$ and $P \in \mathrm{C}(\Sigma, X)$, then $P(t) \preccurlyeq_T t$ always holds. This shows that $\mathrm{SOA}(T)$ is a symbolic ordered algebra. Conversely, if $\mathrm{SOA}(T)$ is a symbolic ordered algebra then $\tau \subseteq \theta_T$ and $P(t) \preccurlyeq_T t$ for all $t \in \mathrm{T}(\Sigma, X)$ and $P \in \mathrm{C}(\Sigma, X)$. Suppose for trees t and t', $\mathrm{c}(t) \subseteq \mathrm{c}(t')$ and $t \in T$ hold. Then there exists a context P, such that $\mathrm{c}(t') = \mathrm{c}(P(t))$. By Lemma 3.19, $t' \,\tau\, P(t)$, and so $t' \;\theta_T\; P(t)$ holds. On the other hand, $P(t) \preccurlyeq_T t$ implies $t' \preccurlyeq_T t$, and this implies $t' \in T$, since $t \in T$. Hence, T is a symbolic tree language by Lemma 3.17. □

Corollary 3.21. *Family SL_Σ is a variety of tree languages and $\mathrm{SL}_\Sigma = \mathbf{SL}(\Sigma)^{\mathtt{t}}$, also family Sym_Σ is a positive variety of tree languages and $\mathrm{Sym}_\Sigma = \mathbf{Sym}(\Sigma)^{\mathtt{t}}$.*

Another characterization of symbolic tree languages is given below. We will show that they are exactly those semilattice languages recognized by so-called translation closed subsets of semilattice algebras.

Proposition 3.22. *For a semilattice algebra $\mathcal{A} = (A, \Sigma)$ the structure $\mathcal{A}_s = (A, \Sigma, \leqslant)$, where $\leqslant$ is defined by*

$$a \leqslant b \iff a = p(b) \text{ for some } p \in \mathrm{Tr}(\mathcal{A})$$

for any $a, b \in A$, is a symbolic ordered algebra.

Proof. It is clear that the relation $\leqslant$ is reflexive and transitive, and it is anti-symmetric by Lemma 2.15. It is also compatible with Σ since for any $a, b \in A$, such that $a \leqslant b$, it follows that $a = p(b)$ for some $p \in \mathrm{Tr}(\mathcal{A})$. Hence $q(a) = q(p(b)) = p(q(b))$, and so $q(a) \leqslant q(b)$ for every $q \in \mathrm{Tr}(\mathcal{A})$. Obviously, $\leqslant$ satisfies $p(a) \leqslant a$ what implies that $\mathcal{A}_s$ is a symbolic ordered algebra. □

Definition 3.23. For an algebra (A, Σ), a subset $D \subseteq A$ is *translation closed* if $d \in D$ implies $p(d) \in D$ for any $p \in \mathrm{Tr}(\mathcal{A})$.

Translation closed subsets are known as *ideals* of algebras, but we have chosen a different name since this notion already has a different meaning here.

Lemma 3.24. *A subset $D \subseteq A$ of a semilattice algebra $\mathcal{A} = (A, \Sigma)$ is translation closed if and only if D is an ideal of the symbolic ordered algebra $\mathcal{A}_s$, where $\mathcal{A}_s$ is defined in Proposition* 3.22.

Proposition 3.25. *A tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is a symbolic tree language if and only if there exist a finite semilattice algebra $\mathcal{A} = (A, \Sigma)$, a morphism $\varphi : \mathcal{T}(\Sigma, X) \to \mathcal{A}$ and a translation closed subset $F \subseteq A$, such that $T = F\varphi^{-1}$.*

4. Generalized positive variety theorem

Generalized varieties of tree languages and generalized varieties of finite algebras were introduced by Steinby [23] who proved a generalized variety theorem for these classes. A variety of finite algebras is a class of finite algebras over a fixed ranked alphabet as the notions of subalgebras, homomorphic images and direct products are defined for algebras over the same ranked alphabet. These notions can be generalized for algebras over different ranked alphabets. A generalized variety of finite algebras is a class of finite algebras over any ranked alphabet that satisfies certain closure properties. Similarly a generalized variety of tree languages is defined. In this section, we generalize our Positive Variety Theorem (Proposition 3.12) to generalized positive varieties of tree languages and generalized varieties of finite ordered algebras. The following definition is the ordered version of Definitions 3.1–3.3, 3.14 in [23].

Definition 4.1. Let $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Omega, \leqslant')$ be ordered algebras.

- The ordered algebra $\mathcal{B}$ is an *order g-subalgebra* of $\mathcal{A}$, in notation $\mathcal{B} \subseteq_g \mathcal{A}$, if $B \subseteq A$, $\Omega_m \subseteq \Sigma_m$ for any $m \geqslant 0$, $f^{\mathcal{B}}$ is the restriction of $f^{\mathcal{A}}$ to B for every $f \in \Omega_m$, and $\leqslant'$ is the restriction of $\leqslant$ on B.
- An *assignment* is a mapping $\kappa : \Sigma \to \Omega$, such that $\kappa(\Sigma_m) \subseteq \Omega_m$ for any $m \geqslant 0$. An *order g-morphism* from $\mathcal{A}$ to $\mathcal{B}$ is a pair (κ, φ) where the mapping $\kappa : \Sigma \to \Omega$ is an assignment and $\varphi : A \to B$ is an order preserving mapping satisfying $f^{\mathcal{A}}(a_1, \ldots, a_m)\varphi = (f\kappa)^{\mathcal{B}}(a_1\varphi, \ldots, a_m\varphi)$ for any $m \geqslant 0$, $f \in \Sigma_m$, and $a_1, \ldots, a_m \in A$. Note that order preserving means that $a \leqslant b$ implies $a\varphi \leqslant' b\varphi$ for all $a, b \in A$. If both κ and φ are surjective, then (κ, φ) is an *order g-epimorphism*, and in that case we write $\mathcal{B} \leftarrow_g \mathcal{A}$ meaning that $\mathcal{B}$ is an *order g-epimorphic image of* $\mathcal{A}$. When $\mathcal{B}$ is an order g-epimorphic image of an order g-subalgebra of $\mathcal{A}$, we write $\mathcal{B} \prec_g \mathcal{A}$. When both κ and φ are bijective and $(\kappa^{-1}, \varphi^{-1})$ is an order g-morphism, (κ, φ) is an *order g-isomorphism*, and $\mathcal{B} \cong_g \mathcal{A}$ means that $\mathcal{B}$ and $\mathcal{A}$ are order g-isomorphic.
- Let $\Sigma^1, \ldots, \Sigma^n$ and Γ be ranked alphabets. The product $\Sigma^1 \times \cdots \times \Sigma^n$ is a ranked alphabet, such that $(\Sigma^1 \times \cdots \times \Sigma^n)_m = \Sigma^1_m \times \cdots \times \Sigma^n_m$ for every $m \geqslant 0$. For any assignment $\kappa : \Gamma \to \Sigma^1 \times \cdots \times \Sigma^n$ and any finite number of ordered algebras $\mathcal{A}_1 =$

$(A_1, \Sigma^1, \leqslant_1), \ldots, \mathcal{A}_n = (A_n, \Sigma^n, \leqslant_n)$, the *$\kappa$-product* of $\mathcal{A}_1, \ldots, \mathcal{A}_n$ is the ordered Γ-algebra $\kappa(\mathcal{A}_1, \ldots, \mathcal{A}_n) = (A_1 \times \cdots \times A_n, \Gamma, \leqslant_1 \times \cdots \times \leqslant_n)$, where the following hold: for any $c \in \Gamma_0$, $f \in \Gamma_m$ $(m > 0)$ and $\mathbf{a}_i = (a_{i1}, \ldots, a_{in}) \in A_1 \times \cdots \times A_n$ $(i \leqslant m)$,

(1) $c^{\kappa(\mathcal{A}_1,\ldots,\mathcal{A}_n)} = (c_1^{\mathcal{A}_1}, \ldots, c_n^{\mathcal{A}_n})$, where $c\kappa = (c_1, \ldots, c_n)$,

(2) $f^{\kappa(\mathcal{A}_1,\ldots,\mathcal{A}_n)}(\mathbf{a}_1, \ldots, \mathbf{a}_m) = (f_1^{\mathcal{A}_1}(a_{11}, \ldots, a_{m1}), \ldots, f_n^{\mathcal{A}_n}(a_{1n}, \ldots, a_{mn}))$, where $f\kappa = (f_1, \ldots, f_n)$, and

(3) $\mathbf{a}_1 \leqslant_1 \times \cdots \times \leqslant_n \mathbf{a}_2 \iff a_{11} \leqslant_1 a_{21} \,\&\, \ldots \,\&\, a_{1n} \leqslant_n a_{2n}$.

Without specifying the assignment κ, such algebras are *g-products*.

A *generalized variety of finite ordered algebras*, a gVFOA for short, is a class $\mathscr{K} = \{\mathscr{K}(\Sigma)\}$ which consists of a class of finite ordered Σ-algebras $\mathscr{K}(\Sigma)$ for any ranked alphabet Σ, and is closed under order g-subalgebras, order g-epimorphic images, and g-products.

Proposition 4.2. *If $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Omega, \leqslant')$ are ordered algebras, $\preccurlyeq$ is a quasi-order on $\mathcal{B}$ and $(\kappa, \varphi) : \mathcal{A} \to \mathcal{B}$ is an order g-morphism, then*

(1) *the image of $\mathcal{A}$, $\mathcal{A}(\kappa, \varphi) = (A\varphi, \Sigma\kappa, \leqslant'')$, where $\leqslant''$ is the restriction of $\leqslant'$ on $A\varphi$, is an order g-subalgebra of $\mathcal{B}$,*

(2) *$\varphi \circ \preccurlyeq \circ\, \varphi^{-1}$ is a quasi-order on $\mathcal{A}$ and $\mathcal{A}/\varphi \circ \preccurlyeq \circ\, \varphi^{-1} \cong_g \mathcal{A}\varphi/\preccurlyeq'$, where $\preccurlyeq'$ is the restriction of $\preccurlyeq$ on $A\varphi$, and*

(3) *if φ is an order g-epimorphism then $\mathcal{A}/\varphi \circ \preccurlyeq \circ\, \varphi^{-1} \cong_g \mathcal{B}/\preccurlyeq$.*

The proof is a direct generalization of that of Proposition 2.3. Also, many of the already presented results have their "generalized" counterparts with slightly different proofs. For example, a result analogous to Proposition 2.8 can be proved. As a corollary, we get that for any g-morphism $(\kappa, \varphi) : \mathcal{T}(\Omega, Y) \to \mathcal{T}(\Sigma, X)$ and tree language $T \subseteq \mathrm{T}(\Sigma, X)$, $\mathrm{SOA}(T\varphi^{-1}) \prec_g \mathrm{SOA}(T)$ holds, and if (κ, φ) is a g-epimorphism then $\mathrm{SOA}(T\varphi^{-1}) \cong_g \mathrm{SOA}(T)$.

Let Σ and Ω be ranked alphabets, X be a leaf alphabet, and $\mathcal{A} = (A, \Omega, \leqslant)$ be an ordered algebra. A tree language $T \subseteq \mathrm{T}(\Sigma, X)$ is *g-recognized* by $\mathcal{A}$ if there exist an ideal $I \trianglelefteq \mathcal{A}$ and an order g-morphism $(\kappa, \varphi) : \mathcal{T}(\Sigma, X) \to \mathcal{A}$ such that $T = I\varphi^{-1}$. Similarly to Proposition 3.3 it can be proved that a tree language T is g-recognized by $\mathcal{A}$ if $\mathrm{SOA}(T) \prec_g \mathcal{A}$. Contrary to Proposition 3.3, the converse of this statement does not hold, for more details see the definition of reduced syntactic algebra in Section 6 of Steinby [23].

Definition 4.3. A *family* of recognizable tree languages $\mathscr{V} = \{\mathscr{V}(\Sigma, X)\}$, where $\mathscr{V}(\Sigma, X)$ consists of recognizable ΣX-tree languages for any ranked alphabet Σ and leaf alphabet X, is a *generalized positive variety of tree languages*, abbreviated by gPVTL, if it is closed under positive Boolean operations (intersections and unions), inverse translations, and inverse g-morphisms.

Definition 4.4. Let $\mathscr{K} = \{\mathscr{K}(\Sigma)\}$ be a gVFOA. Define the family $\mathscr{K}^{\mathtt{t}} = \{\mathscr{K}^{\mathtt{t}}(\Sigma, X)\}$ to be the family of tree languages whose syntactic ordered algebras are in $\mathscr{K}$, that is $\mathscr{K}^{\mathtt{t}}(\Sigma, X) = \{T \subseteq \mathrm{T}(\Sigma, X) \mid \mathrm{SOA}(T) \in \mathscr{K}(\Sigma)\}$.

For a gPVTL $\mathscr{V} = \{\mathscr{V}(\Sigma, X)\}$, let $\mathscr{V}^{\mathtt{a}} = \{\mathscr{V}^{\mathtt{a}}(\Sigma)\}$ be the gVFOA generated by the class $\{\mathrm{SOA}(T) \mid T \in \mathscr{V}(\Sigma, X)$ for some $\Sigma, X\}$.

It can be proved similarly to Lemmas 3.7, 3.10 and Corollary 3.9 that every gVFOA is generated by syntactic ordered algebras of some tree languages and that if every tree language recognizable by a finite ordered algebra $\mathcal{A}$ belongs to a gPVTL $\mathscr{V}$ then $\mathcal{A} \in \mathscr{V}^{\mathtt{a}}$.

Proposition 4.5 (*Generalized Positive Variety Theorem*). *The operations $\mathscr{K} \mapsto \mathscr{K}^{\mathtt{t}}$ and $\mathscr{V} \mapsto \mathscr{V}^{\mathtt{a}}$ are mutually inverse lattice isomorphisms between the class of all gVFOA's and the class of gPVTL's, i.e., $\mathscr{V}^{\mathtt{at}} = \mathscr{V}$ and $\mathscr{K}^{\mathtt{ta}} = \mathscr{K}$.*

Proof. The facts that for a gVFOA $\mathscr{K}$ the family $\mathscr{K}^{\mathtt{t}}$ is a gPVTL and that the mappings $\mathscr{K} \mapsto \mathscr{K}^{\mathtt{t}}$ and $\mathscr{V} \mapsto \mathscr{V}^{\mathtt{a}}$ are monotone, as well as the relations $\mathscr{V} \subseteq \mathscr{V}^{\mathtt{at}}$ and $\mathscr{K}^{\mathtt{ta}} = \mathscr{K}$, can be proved in a way similar to the proofs of the corresponding claims in Section 3.2. We are proving here only the inclusion $\mathscr{V}^{\mathtt{at}} \subseteq \mathscr{V}$.

Suppose $T \in \mathscr{V}^{\mathtt{at}}(\Sigma, X)$. There exist some ranked alphabets $\Sigma^1, \ldots, \Sigma^n$, leaf alphabets $X_1, \ldots, X_n$ and tree languages $T_1 \in \mathscr{V}(\Sigma^1, X_1), \ldots, T_n \in \mathscr{V}(\Sigma^n, X_n)$ such that $\mathrm{SOA}(T) \prec_g \kappa(\mathrm{SOA}(T_1), \ldots, \mathrm{SOA}(T_n))$ where $\kappa : \Gamma \to \Sigma^1 \times \cdots \times \Sigma^n$ is an assignment for a ranked alphabet Γ. Let $\mathcal{A}_j = \mathrm{SOA}(T_j)$ for $j \leqslant n$. Then T is g-recognized by $\kappa(\mathcal{A}_1, \ldots, \mathcal{A}_n)$, and so there exist an order g-morphism $(\lambda, \varphi) : \mathcal{T}(\Sigma, X) \to \kappa(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ and an ideal $I \trianglelefteq \kappa(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ such that $T = I\varphi^{-1}$. Let $\varphi_j : \mathrm{T}(\Sigma, X) \to A_j$ be the composition of φ with the jth projection function $\prod_{i=1}^n A_i \to A_j$, and $\lambda_j : \Sigma \to \Sigma^j$ be the composition of $\lambda\kappa : \Sigma \to \Sigma^1 \times \cdots \times \Sigma^n$ with the jth projection $\Sigma^1 \times \cdots \times \Sigma^n \to \Sigma^j$. Then $(\lambda_j, \varphi_j) : \mathcal{T}(\Sigma, X) \to \mathcal{A}_j$ is an order g-morphism, and similarly to the proof of Lemma 3.11, $T = I\varphi^{-1} = \bigcup_{\mathbf{a} \in I} (\mathbf{a}]\varphi^{-1} = \bigcup_{(a_1, \ldots, a_n) \in I} \bigcap_{j \leqslant n} (a_j]\varphi_j^{-1}$.

For showing $T \in \mathscr{V}(\Sigma, X)$ it suffices to show $(a_j]\varphi_j^{-1} \in \mathscr{V}(\Sigma, X)$ for every $j \leqslant n$. Fix a $j \leqslant n$. Let $\varphi_{T_j} : \mathcal{T}(\Sigma^j, X_j) \to \mathcal{A}_j$ be the syntactic morphism of T_j. A g-morphism $(\lambda_j, \chi_j) : \mathcal{T}(\Sigma, X) \to \mathcal{T}(\Sigma^j, X_j)$ such that $\chi_j \varphi_{T_j} = \varphi_j$ can be constructed. Then $(a_j]\varphi_j^{-1} = (a_j]\varphi_{T_j}^{-1}\chi_j^{-1}$, and since $\mathscr{V}$ is closed under inverse g-morphisms, for showing $(a_j]\varphi_j^{-1} \in \mathscr{V}(\Sigma, X)$ it is enough to show $(a_j]\varphi_{T_j}^{-1} \in \mathscr{V}(\Sigma^j, X_j)$. It was shown in the proof of Lemma 3.11 that $(a_j]\varphi_{T_j}^{-1} = \bigcap\{P^{-1}(T_j) \mid P \in \mathrm{C}(\Sigma^j, X_j), P(t) \in T_j\}$ for some $t \in \mathrm{T}(\Sigma^j, X_j)$. Hence, from $T_j \in \mathscr{V}(\Sigma^j, X_j)$ and the fact that $\mathscr{V}$ is closed under inverse translations and positive Boolean operations, it follows that $(a_j]\varphi_{T_j}^{-1} \in \mathscr{V}(\Sigma^j, X_j)$. Therefore, $(a_j]\varphi_j^{-1} \in \mathscr{V}(\Sigma, X)$ for any j, thus $T \in \mathscr{V}(\Sigma, X)$. $\square$

4.1. Examples

The examples of families of recognizable tree languages and classes of finite ordered algebras in the previous sections do not heavily depend on their ranked alphabets. Here, we will see that the collection of those varieties for various ranked alphabets form generalized varieties.

Let $\mathbf{Nil} = \{\mathbf{Nil}(\Sigma)\}$ be the class of all ordered nilpotent algebras for every ranked alphabet Σ, and $\mathrm{Cof} = \{\mathrm{Cof}(\Sigma, X)\}$ be the family of all cofinite tree languages for all ranked alphabets Σ and leaf alphabets X.

Proposition 4.6. *Class* **Nil** *is a gVFOA, family* Cof *is a gPVTL, and* Cof $=$ $\mathbf{Nil}^{\mathtt{t}}$.

That Cof is a gPVTL can be verified directly: the family is closed under positive Boolean operations, inverse translations and inverse g-morphisms. Similarly, **Nil** can be proved to be a gVFOA. From Proposition 3.14 it follows that $T \in \mathrm{Cof}(\Sigma, X)$ iff $\mathrm{SOA}(T) \in \mathbf{Nil}(\Sigma)$ for any $T \subseteq \mathrm{T}(\Sigma, X)$, which implies that $\mathrm{Cof} = \mathbf{Nil}^{\mathtt{t}}$.

Let $\mathbf{SL} = \{\mathbf{SL}(\Sigma)\}$ and $\mathbf{Sym} = \{\mathbf{Sym}(\Sigma)\}$ be, respectively, the classes of all semilattice algebras and symbolic ordered algebras for every ranked alphabet Σ, and $\mathrm{SL} = \{\mathrm{SL}(\Sigma, X)\}$ and $\mathrm{Sym} = \{\mathrm{Sym}(\Sigma, X)\}$ be, respectively, the families of all semilattice and symbolic tree languages for all ranked alphabets Σ and leaf alphabets X.

Proposition 4.7. (1) *Class* **SL** *is a generalized variety of finite algebras, family* SL *is a generalized variety of recognizable tree languages, and* $\mathrm{SL} = \mathbf{SL}^{\mathtt{t}}$.

(2) *Class* **Sym** *is a gVFOA, family* Sym *is a gPVTL, and* $\mathrm{Sym} = \mathbf{Sym}^{\mathtt{t}}$.

5. Definability by ordered monoids

An important class of ordered algebras is the class of ordered monoids. Let us recall that an ordered monoid is a structure $\mathcal{M} = (M, \cdot, \lesssim)$ where $(M, \cdot)$ is a monoid and $\lesssim$ is an order on M compatible with $\cdot$ (called "stable order'' in [13]), i.e., for any $a, b, m, m' \in M$ if $a \lesssim b$ then $m \cdot a \cdot m' \lesssim m \cdot b \cdot m'$.

5.1. Ordered algebras definable by ordered monoids

Translations of ordered algebras can be ordered as follows:

Definition 5.1. The *ordered translation monoid* of an ordered algebra $\mathcal{A}$ is the structure $\mathrm{OTr}(\mathcal{A}) = (\mathrm{Tr}(\mathcal{A}), \cdot, \lesssim_{\mathcal{A}})$, where $(\mathrm{Tr}(\mathcal{A}), \cdot)$ is the translation monoid of $\mathcal{A}$ and the binary relation $\lesssim_{\mathcal{A}}$ is defined on $\mathrm{Tr}(\mathcal{A})$ by

$$p \lesssim_{\mathcal{A}} q \iff (\forall a \in A)\big(p(a) \leqslant q(a)\big)$$

for $p, q \in \mathrm{Tr}(\mathcal{A})$.

The relation $\lesssim_{\mathcal{A}}$ is indeed an order on $\mathrm{Tr}(\mathcal{A})$ compatible with the composition of translations: if $p \lesssim_{\mathcal{A}} q$ then $p \cdot r \lesssim_{\mathcal{A}} q \cdot r$ and $r \cdot p \lesssim_{\mathcal{A}} r \cdot q$ for any $p, q, r \in \mathrm{Tr}(\mathcal{A})$.

The following proposition is the ordered version of Steinby [23, Lemma 10.7].

Proposition 5.2. *For any finite ordered algebras* $\mathcal{A}$ *and* $\mathcal{B}$,

(1) *if* $\mathcal{A} \subseteq_g \mathcal{B}$ *then* $\mathrm{OTr}(\mathcal{A}) \prec \mathrm{OTr}(\mathcal{B})$;

(2) *if* $\mathcal{A} \leftarrow_g \mathcal{B}$ *then* $\mathrm{OTr}(\mathcal{A}) \leftarrow \mathrm{OTr}(\mathcal{B})$;

(3) $\mathrm{OTr}(\kappa(\mathcal{A}, \mathcal{B})) \subseteq \mathrm{OTr}(\mathcal{A}) \times \mathrm{OTr}(\mathcal{B})$ *for any g-product* $\kappa(\mathcal{A}, \mathcal{B})$.

Proof. Let $\mathcal{A} = (A, \Sigma, \leqslant)$ and $\mathcal{B} = (B, \Omega, \leqslant')$.

(1) Let $\mathcal{M}$ be the order submonoid of $\mathrm{OTr}(\mathcal{B})$ generated by the elementary translations of the form $f^{\mathcal{B}}(a_1, \ldots, \xi, \ldots, a_m)$ for any $f \in \Sigma_m$ $(m > 0)$ and $a_1, \ldots, a_m \in A$. The

mapping $f^{\mathcal{B}}(a_1, \ldots, \xi, \ldots, a_m) \mapsto f^{\mathcal{A}}(a_1, \ldots, \xi, \ldots, a_m)$ can be uniquely extended to an order epimorphism $\mathcal{M} \to \mathrm{OTr}(\mathcal{A})$. Thus $\mathrm{OTr}(\mathcal{A}) \leftarrow \mathcal{M} \subseteq \mathrm{OTr}(\mathcal{B})$.

(2) Suppose $(\kappa, \varphi) : \mathcal{B} \to \mathcal{A}$ is an order g-epimorphism. By a generalized counterpart of Proposition 2.6, the mapping (κ, φ) induces a monoid epimorphism $\mathrm{Tr}(\mathcal{A}) \to \mathrm{Tr}(\mathcal{B})$, $p \mapsto p_{(\kappa,\varphi)}$, such that $p(a)\varphi = p_{(\kappa,\varphi)}(a\varphi)$ for $a \in A$. It also preserves the translation order. Indeed, for any $p, q \in \mathrm{OTr}(\mathcal{B})$, from $p \lesssim_{\mathcal{B}} q$ follows that $p(b) \leqslant' q(b)$ for any $b \in B$, what further implies $p(b)\varphi \leqslant q(b)\varphi$, and so $p_{(\kappa,\varphi)}(b\varphi) \leqslant q_{(\kappa,\varphi)}(b\varphi)$ for any $b \in B$. This gives $p_{(\kappa,\varphi)}(a) \leqslant q_{(\kappa,\varphi)}(a)$ for any $a \in A$, and so $p_{(\kappa,\varphi)} \lesssim_{\mathcal{A}} q_{(\kappa,\varphi)}$.

(3) Let Γ be a ranked alphabet and $\kappa : \Gamma \to \Sigma \times \Omega$ be an assignment. It is easy to verify that the mapping

$$g^{\kappa(\mathcal{A},\mathcal{B})}\big((a_1, b_1), \ldots, \xi, \ldots, (a_m, b_m)\big) \mapsto \Big(f^{\mathcal{A}}(a_1, \ldots, \xi, \ldots, a_m), h^{\mathcal{B}}(b_1, \ldots, \xi, \ldots, b_m)\Big)$$

for $a_1, \ldots, a_m \in A, b_1, \ldots, b_m \in B$ and $g \in \Gamma_m$ $(m > 0)$, where $g\kappa = (f, h)$, can be extended to a monomorphism $\psi : \mathrm{OTr}(\kappa(\mathcal{A}, \mathcal{B})) \to \mathrm{OTr}(\mathcal{A}) \times \mathrm{OTr}(\mathcal{B})$ which satisfies $p(a, b) = (p\psi_1(a), p\psi_2(b))$ for all $a \in A$, $b \in B$ and $p \in \mathrm{Tr}(\kappa(\mathcal{A}, \mathcal{B}))$, where ψ_1 and ψ_2 are the components of ψ, i.e., $p\psi = (p\psi_1, p\psi_2)$. The mapping ψ is also order preserving. Indeed, for $p, q \in \mathrm{Tr}(\kappa(\mathcal{A}, \mathcal{B}))$, such that $p \lesssim_{\kappa(\mathcal{A},\mathcal{B})} q$, i.e., $p(a, b) \leqslant \times \leqslant' q(a, b)$ for all $a \in A, b \in B$, it follows $p\psi_1(a) \leqslant q\psi_1(a)$ and $p\psi_2(b) \leqslant' q\psi_2(b)$ for all $a \in A, b \in B$, what means $p\psi_1 \lesssim_{\mathcal{A}} q\psi_1$ and $p\psi_2 \lesssim_{\mathcal{B}} q\psi_2$, and so $(p\psi_1, p\psi_2) \lesssim_{\mathcal{A}} \times \lesssim_{\mathcal{B}} (q\psi_1, q\psi_2)$, i.e., $p\psi \lesssim_{\mathcal{A}} \times \lesssim_{\mathcal{B}} q\psi$. $\square$

Definition 5.3. A *variety of finite ordered monoids*, in notation VFOM, is a class of finite ordered monoids closed under order submonoids, order epimorphic images and finite direct products.

For a VFOM $\mathbf{M}$, $\mathbf{M}^{\mathsf{a}}$ is the class of all finite ordered algebras whose ordered translation monoids are in $\mathbf{M}$, i.e.,

$$\mathbf{M}^{\mathsf{a}} = \{\mathcal{A} \mid \mathcal{A} \text{ is an ordered algebra such that } \mathrm{OTr}(\mathcal{A}) \in \mathbf{M}\}.$$

A class of finite ordered algebras $\mathscr{K}$ is said to be *definable by ordered translation monoids* if there is a VFOM $\mathbf{M}$, such that $\mathbf{M}^{\mathsf{a}} = \mathscr{K}$.

The next result follows from Proposition 5.2.

Corollary 5.4. *For any VFOM* $\mathbf{M}$, *the class* $\mathbf{M}^{\mathsf{a}}$ *is a gVFOA.*

It is well-known that not every gVFOA is definable by syntactic ordered monoids. In this section, we give necessary and sufficient conditions for a class of algebras to be of the form $\mathbf{M}^{\mathsf{a}}$ for some VFOA $\mathbf{M}$.

Definition 5.5. For any set D, let $\Lambda_D = \{\overline{d} \mid d \in D\}$ be the unary ranked alphabet consisting of unary function symbols $\overline{d}$ for each $d \in D$. For a finite ordered monoid $\mathcal{M} = (M, \cdot, \lesssim)$ the unary ordered algebra $\mathcal{M}^{\mathrm{v}} = (M, \Lambda_M, \lesssim)$ is defined by $\overline{m}^{\mathcal{M}^{\mathrm{v}}}(a) = a \cdot m$ for all $a, m \in M$.

The structure $\mathcal{M}^v$ for a finite ordered monoid $\mathcal{M}$ is indeed an ordered algebra since for any $a, b, m \in M$,

$$a \lesssim b \ \Rightarrow\ a \cdot m \lesssim b \cdot m \ \Rightarrow\ \overline{m}^{\mathcal{M}^v}(a) \lesssim \overline{m}^{\mathcal{M}^v}(b).$$

Proposition 5.6. *For a finite ordered monoid $\mathcal{M} = (M, \cdot, \lesssim)$,*

$$\mathrm{OTr}(\mathcal{M}^v) \cong \mathcal{M}.$$

Proof. For the sake of simplicity, operations of $\mathcal{M}^v$ are denoted by $\overline{m}$ instead of $\overline{m}^{\mathcal{M}^v}$. Elementary translations of $\mathcal{M}^v$ are of the form $\overline{m}(\xi)$ where $m \in M$, and clearly $\overline{m}(\xi) \cdot \overline{m'}(\xi) = \overline{m \cdot m'}(\xi)$ for all $m, m' \in M$. For the unit element 1_M of $\mathcal{M}$, the translation $\overline{1_M}(\xi)$ is the identity translation of $\mathcal{M}^v$. This means that $\mathrm{Tr}(\mathcal{M}^v) = \{\overline{m}(\xi) \mid m \in M\}$. Moreover, $\overline{m}(\xi) \neq \overline{m'}(\xi)$ whenever $m \neq m'$, since $\overline{m}(\xi) = \overline{m'}(\xi)$ implies $m = 1_M \cdot m = \overline{m}(1_M) = \overline{m'}(1_M) = 1_M \cdot m' = m'$. Hence, the mapping $\mathcal{M} \to \mathrm{OTr}(\mathcal{M}^v)$, $m \mapsto \overline{m}(\xi)$ is a monoid isomorphism. It is also an order isomorphism. Indeed, for any $m, m' \in M$, $m \lesssim m'$ iff $a \cdot m \lesssim a \cdot m'$ for any $a \in M$, i.e., $\overline{m}(a) \lesssim \overline{m'}(a)$ for any $a \in M$, what is, by definition, equivalent to $\overline{m}(\xi) \lesssim_{\mathcal{M}^v} \overline{m'}(\xi)$. $\square$

Proposition 5.7. *For all finite ordered monoids $\mathcal{M}$ and $\mathcal{P}$,*
(1) *if $\mathcal{M} \subseteq \mathcal{P}$ then $\mathcal{M}^v \subseteq_g \mathcal{P}^v$;*
(2) *if $\mathcal{M} \leftarrow \mathcal{P}$ then $\mathcal{M}^v \leftarrow_g \mathcal{P}^v$;*
(3) *$(\mathcal{M} \times \mathcal{P})^v \cong_g \kappa(\mathcal{M}^v, \mathcal{P}^v)$ for some g-product $\kappa(\mathcal{M}^v, \mathcal{P}^v)$.*

Proof. Assume $\mathcal{M} = (M, \cdot, \lesssim)$ and $\mathcal{P} = (P, \cdot, \lesssim')$. The statement (1) is obvious. For (2) we note that if $\varphi : \mathcal{P} \to \mathcal{M}$ is an order monoid epimorphism, then $(\overline{\varphi}, \varphi) : \mathcal{P}^v \to \mathcal{M}^v$, where $\overline{\varphi} : \Lambda_P \to \Lambda_M$ is defined by $(\overline{m})\overline{\varphi} = \overline{m\varphi}$, is an order g-epimorphism. For proving (3) define the assignment $\kappa : \Lambda_{M \times P} \to \Lambda_M \times \Lambda_P$ by $\overline{(m, p)}\kappa = (\overline{m}, \overline{p})$ for $m \in M$, $p \in P$, and let $\kappa(\mathcal{M}^v, \mathcal{P}^v)$ be the corresponding g-product of $\mathcal{M}^v$ and $\mathcal{P}^v$. It is easy to verify that the mappings $(\lambda, \varphi) : (\mathcal{M} \times \mathcal{P})^v \to \kappa(\mathcal{M}^v, \mathcal{P}^v)$, where λ is the identity mapping on $\Lambda_{M \times P}$ and φ is the identity mapping on $M \times P$, is an order g-isomorphism. $\square$

The clause (3) of Proposition 5.7 can be generalized to any finite number of finite ordered monoids $\mathcal{M}_1, \ldots, \mathcal{M}_n$, i.e., $(\mathcal{M}_1 \times \cdots \times \mathcal{M}_n)^v \cong_g \kappa(\mathcal{M}_1^v, \ldots, \mathcal{M}_n^v)$ for some g-product $\kappa(\mathcal{M}_1^v, \ldots, \mathcal{M}_n^v)$.

Definition 5.8. For a finite ordered algebra $\mathcal{A}$, the unary algebra $\mathcal{A}^\rho$ is defined to be $\big(\mathrm{OTr}(\mathcal{A})\big)^v$.

Corollary 5.9. *If $\mathrm{OTr}(\mathcal{A}) \prec \mathrm{OTr}(\mathcal{A}_1) \times \cdots \times \mathrm{OTr}(\mathcal{A}_n)$ holds for finite ordered algebras $\mathcal{A}, \mathcal{A}_1, \ldots, \mathcal{A}_n$ $(n > 0)$, then $\mathcal{A}^\rho \prec_g \kappa(\mathcal{A}_1^\rho, \ldots, \mathcal{A}_n^\rho)$ for some g-product $\kappa(\mathcal{A}_1^\rho, \ldots, \mathcal{A}_n^\rho)$.*

This is an immediate consequence of Proposition 5.7.

Our characterization of gVFOA's definable by syntactic ordered monoids is the following.

Proposition 5.10. *For a class $\mathscr{K}$ of finite ordered algebras the following conditions are equivalent*:

(1) *$\mathscr{K}$ is definable by ordered translation monoids*;

(2) *$\mathscr{K}$ is a gVFOA, such that for all finite ordered algebras $\mathcal{A}$ and $\mathcal{B}$, if* $\mathrm{OTr}(\mathcal{A}) \cong \mathrm{OTr}(\mathcal{B})$ *and $\mathcal{A} \in \mathscr{K}$ then $\mathcal{B} \in \mathscr{K}$*;

(3) *$\mathscr{K}$ is a gVFOA, such that $\mathcal{A} \in \mathscr{K} \iff \mathcal{A}^{\rho} \in \mathscr{K}$ for any $\mathcal{A}$.*

Proof. Implication (1) $\Rightarrow$ (2) is obvious, and (2) $\Rightarrow$ (3) follows from Proposition 5.6. For (3) $\Rightarrow$ (1), suppose that a gVFOA $\mathscr{K}$ satisfies the equivalence $\mathcal{A} \in \mathscr{K} \Leftrightarrow \mathcal{A}^{\rho} \in \mathscr{K}$ for any finite ordered algebra $\mathcal{A}$. Let $\mathbf{M}$ be the VFOM generated by $\{\mathrm{OTr}(\mathcal{A}) \mid \mathcal{A} \in \mathscr{K}\}$. We are showing that $\mathscr{K} = \mathbf{M}^{\mathsf{a}}$. Obviously, the inclusion $\mathscr{K} \subseteq \mathbf{M}^{\mathsf{a}}$ holds. For the opposite inclusion, let $\mathcal{B} \in \mathbf{M}^{\mathsf{a}}$. So, $\mathrm{OTr}(\mathcal{B}) \prec \mathrm{OTr}(\mathcal{A}_1) \times \cdots \times \mathrm{OTr}(\mathcal{A}_n)$ for some $\mathcal{A}_1, \ldots, \mathcal{A}_n \in \mathscr{K}$. By Corollary 5.9, $\mathcal{B}^{\rho} \prec_g \kappa(\mathcal{A}_1^{\rho}, \ldots, \mathcal{A}_n^{\rho})$ for some g-product $\kappa(\mathcal{A}_1^{\rho}, \ldots, \mathcal{A}_n^{\rho})$. Since $\mathcal{A}_1^{\rho}, \ldots, \mathcal{A}_n^{\rho} \in \mathscr{K}$ then $\mathcal{B}^{\rho} \in \mathscr{K}$, and hence $\mathcal{B} \in \mathscr{K}$. Thus $\mathbf{M}^{\mathsf{a}} \subseteq \mathscr{K}$. $\square$

Remark 5.11. Proposition 5.7 and the proof of Proposition 5.10 also yield the fact that for any gVFOA $\mathscr{K}$ definable by ordered translation monoids, the class $\{\mathrm{OTr}(\mathcal{A}) \mid \mathcal{A} \in \mathscr{K}\}$ is a variety of finite ordered monoids.

5.2. Examples

A semigroup with zero is *n-nilpotent*, $n \in \mathbb{N}$, if product of any n elements is zero, and it is *nilpotent* if it is n-nilpotent for some $n \in \mathbb{N}$.

Lemma 5.12. *If $\mathcal{A} = (A, \Sigma, \leqslant)$ is an ordered n-nilpotent algebra, then the ordered translation semigroup* $\mathrm{OTrS}(\mathcal{A}) = (\mathrm{TrS}(\mathcal{A}), \cdot, \lesssim_{\mathcal{A}})$ *is a nilpotent semigroup where zero element is the least element.*

Proof. Since $p_1 \cdots p_n(a) \leqslant q_1 \cdots q_n(a) \leqslant p_1 \cdots p_n(a)$ for every $a \in A$, it follows that $p_1 \cdots p_n = q_1 \cdots q_n$ for all $p_1, \ldots, p_n, q_1, \ldots, q_n \in \mathrm{TrS}(\mathcal{A})$. Therefore, $p_1 \cdots p_n \in \mathrm{TrS}(\mathcal{A})$ is the zero element of $\mathrm{TrS}(\mathcal{A})$ and it is n-nilpotent. Moreover, $p_1 \cdots p_n(a) \leqslant q(a)$ holds for all $q \in \mathrm{TrS}(\mathcal{A})$ and $a \in A$, and so $p_1 \cdots p_n \lesssim_{\mathcal{A}} q$. Hence, zero is the least element in $\mathrm{TrS}(\mathcal{A})$. $\square$

The converse of Lemma 5.12 does not hold. Indeed, let $\Lambda = \Lambda_1 = \{f\}$ and $A = \{a, b\}$, $B = \{a, b, c\}$. Define the ordered Λ-algebras $\mathcal{A} = (A, \Lambda, \leqslant)$ and $\mathcal{B} = (B, \Lambda, \leqslant')$ by $f^{\mathcal{A}}(a) = f^{\mathcal{A}}(b) = b$, $f^{\mathcal{B}}(a) = f^{\mathcal{B}}(b) = b$, $f^{\mathcal{B}}(c) = c$, and $\leqslant\, = \{(a, a), (b, a), (b, b)\}$, $\leqslant' = \{(a, a), (b, a), (b, b), (c, c)\}$. Then the ordered translation semigroups of $\mathcal{A}$ and $\mathcal{B}$ are the trivial one-element semigroups, while $\mathcal{A}$ is an ordered nilpotent algebra and $\mathcal{B}$ is not. Hence, **Nil** is not definable by ordered translation monoids or semigroups.

By Lemma 2.14 class **SL** is definable by semilattice monoids.

An ordered monoid $\mathcal{M} = (M, \cdot, \lesssim)$ is *symbolic* if it is a semilattice monoid and the unit 1_M is the greatest element of the monoid, i.e., $m \lesssim 1_M$ for every $m \in M$.

Lemma 5.13. *An ordered algebra is symbolic if and only if its ordered translation monoid is symbolic.*

Proof. It is easy to see that an ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$ is symbolic if and only if (A, Σ) is a semilattice algebra and $p(a) \leqslant a$ holds for all $a \in A$ and $p \in \mathrm{Tr}(\mathcal{A})$, what is equivalent to $p \lesssim_{\mathcal{A}} 1_A$. Thus, from Lemma 2.14, it follows that $\mathcal{A}$ is symbolic if and only if $\mathrm{OTr}(\mathcal{A})$ is a symbolic ordered monoid. $\square$

Therefore, class **Sym** is definable by ordered translation monoids.

5.3. Tree languages definable by ordered monoids

Let Σ be a ranked alphabet and X be a leaf alphabet.

Definition 5.14. For any tree language $T \subseteq \mathrm{T}(\Sigma, X)$, the quasi-order $\precsim_T$ is defined on ΣX-contexts by the following: for $P, Q \in \mathrm{C}(\Sigma, X)$,

$$P \precsim_T Q \iff (\forall R \in \mathrm{C}(\Sigma, X))(\forall t \in \mathrm{T}(\Sigma, X))\big(t \cdot Q \cdot R \in T \;\Rightarrow\; t \cdot P \cdot R \in T\big).$$

We note that the equivalence relation of $\precsim_T$ is the m-congruence of T [23]:

$$P \mu_T Q \iff (\forall R \in \mathrm{C}(\Sigma, X))(\forall t \in \mathrm{T}(\Sigma, X))\big(t \cdot P \cdot R \in T \;\Leftrightarrow\; t \cdot Q \cdot R \in T\big).$$

The quotient monoid $(\mathrm{C}(\Sigma, X)/\mu_T, \cdot)$ is called the syntactic monoid of T.

The *syntactic ordered monoid* of T is $\mathrm{SOM}(T) = (\mathrm{C}(\Sigma, X)/\mu_T, \cdot, \lesssim_T)$, where $\lesssim_T$ is the order induced by $\precsim_T$:

$$P/\mu_T \lesssim_T Q/\mu_T \;\Leftrightarrow\; P \precsim_T Q$$

for $P, Q \in \mathrm{C}(\Sigma, X)$; cf. [23] or [25]. It is easy to verify that the relation $P \precsim_T Q$ implies $R \cdot P \cdot S \precsim_T R \cdot Q \cdot S$ for any $P, Q, R, S \in \mathrm{C}(\Sigma, X)$. Thus, the structure $\mathrm{SOM}(T)$ is indeed an ordered monoid.

It is known that the syntactic monoid of a tree language is the translation monoid of the syntactic algebra of the language ([18,23]). The following is the corresponding proposition for ordered translation monoids and syntactic ordered algebras.

Proposition 5.15. *For a tree language* $T \subseteq \mathrm{T}(\Sigma, X)$,

$$\mathrm{OTr}(\mathrm{SOA}(T)) \cong \mathrm{SOM}(T).$$

Proof. It is easy to see that the mapping

$$f(t_1, \ldots, \xi, \ldots, t_m) \mapsto f^{\mathrm{SOA}(T)}(t_1/\theta_T, \ldots, \xi, \ldots, t_m/\theta_T)$$

can be extended to a monoid epimorphism $\varphi : \mathrm{C}(\Sigma, X) \to \mathrm{OTr}(\mathrm{SOA}(T))$ which satisfies $P\varphi(t/\theta_T) = (t \cdot P)/\theta_T$ for all $t \in \mathrm{T}(\Sigma, X)$, $P \in \mathrm{C}(\Sigma, X)$. We are proving that for any $P, Q \in \mathrm{C}(\Sigma, X)$, $P \precsim_T Q$ iff $P\varphi \lesssim_{\mathrm{SOA}(T)} Q\varphi$. Indeed, $P \precsim_T Q$ means by definition that $t \cdot Q \cdot R \in T$ implies $t \cdot P \cdot R \in T$ for all $t \in \mathrm{T}(\Sigma, X)$, $R \in \mathrm{C}(\Sigma, X)$, i.e., $t \cdot P \preccurlyeq_T t \cdot Q$ for every $t \in \mathrm{T}(\Sigma, X)$, or equivalently, $(t \cdot P)/\theta_T \leqslant_T (t \cdot Q)/\theta_T$ for every $t \in \mathrm{T}(\Sigma, X)$. This is further equivalent to $P\varphi(t/\theta_T) \leqslant_T Q\varphi(t/\theta_T)$ for every $t \in \mathrm{T}(\Sigma, X)$, or in other words, $P\varphi \lesssim_{\mathrm{SOA}(T)} Q\varphi$. Thus $\varphi \circ \lesssim_{\mathrm{SOA}(T)} \circ\, \varphi^{-1} = \precsim_T$, and then, from Proposition 2.3, it follows that $\mathrm{SOM}(T) \cong \mathrm{OTr}(\mathrm{SOA}(T))$. $\square$

The following is implied by Corollary 3.4 and Propositions 5.2 and 5.15.

Corollary 5.16. *For ranked alphabets Σ and Ω, leaf alphabets X and Y, a ΣX-context $P \in \mathrm{C}(\Sigma, X)$, an order g-morphism $(\kappa, \varphi) : \mathcal{T}(\Omega, Y) \to \mathcal{T}(\Sigma, X)$, and tree languages $T, T' \subseteq \mathrm{T}(\Sigma, X)$,*
(1) $\mathrm{SOM}(T \cap T'), \mathrm{SOM}(T \cup T') \prec \mathrm{SOM}(T) \times \mathrm{SOM}(T')$;
(2) $\mathrm{SOM}(P^{-1}(T)) \leftarrow \mathrm{SOM}(T)$;
(3) $\mathrm{SOM}(T\varphi^{-1}) \prec \mathrm{SOM}(T)$ *and if* (κ, φ) *is a g-epimorphism then* $\mathrm{SOM}(T\varphi^{-1}) \cong \mathrm{SOM}(T)$.

Definition 5.17. For a VFOM $\mathbf{M}$, let $\mathbf{M}^{\mathtt{t}}$ be the family of all recognizable tree languages whose syntactic ordered monoids are in $\mathbf{M}$, that is to say, for any tree language $T \subseteq \mathrm{T}(\Sigma, X)$, $T \in \mathbf{M}^{\mathtt{t}}(\Sigma, X) \Leftrightarrow \mathrm{SOM}(T) \in \mathbf{M}$.

A family of recognizable tree languages $\mathscr{V}$ is *definable by syntactic ordered monoids* if there is a VFOM $\mathbf{M}$ such that $\mathbf{M}^{\mathtt{t}} = \mathscr{V}$.

By Corollary 5.16, the family $\mathbf{M}^{\mathtt{t}}$ is a gPVTL for any VFOM $\mathbf{M}$. In this subsection, we characterize the gPVTL's that are definable by syntactic ordered monoids.

Lemma 5.18. *For any VFOM* $\mathbf{M}$ *the following hold*:
(1) $\mathbf{M}^{\mathtt{at}} = \mathbf{M}^{\mathtt{t}}$; (2) $\mathbf{M}^{\mathtt{ta}} = \mathbf{M}^{\mathtt{a}}$.

Proof. (1) For any tree language $T \subseteq \mathrm{T}(\Sigma, X)$, by Proposition 5.15,
$T \in \mathbf{M}^{\mathtt{at}}(\Sigma, X) \Leftrightarrow \mathrm{SOA}(T) \in \mathbf{M}^{\mathtt{a}} \Leftrightarrow \mathrm{OTr}(\mathrm{SOA}(T)) \in \mathbf{M} \Leftrightarrow \mathrm{SOM}(T) \in \mathbf{M} \Leftrightarrow T \in \mathbf{M}^{\mathtt{t}}(\Sigma, X)$.

(2) By (1) and Proposition 4.5, $(\mathbf{M}^{\mathtt{t}})^{\mathtt{a}} = (\mathbf{M}^{\mathtt{at}})^{\mathtt{a}} = (\mathbf{M}^{\mathtt{a}})^{\mathtt{ta}} = \mathbf{M}^{\mathtt{a}}$. $\square$

Corollary 5.19. (1) *A gPVTL $\mathscr{V}$ is definable by syntactic ordered monoids if and only if $\mathscr{V}^{\mathtt{a}}$ is a gVFOA definable by ordered translation monoids.*

(2) *A gVFOA $\mathscr{K}$ is definable by ordered translation monoids if and only if $\mathscr{K}^{\mathtt{t}}$ is a gPVTL definable by syntactic ordered monoids.*

Definition 5.20. Let Σ, Ω be ranked alphabets and X, Y be leaf alphabets. A *tree homomorphism* is a mapping $\varphi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, Y)$ determined by some mappings $\varphi_X : X \to \mathrm{T}(\Omega, Y)$ and $\varphi_m : \Sigma_m \to \mathrm{T}(\Omega, Y \cup \{\xi_1, \ldots, \xi_m\})$, where $\Sigma_m \neq \emptyset$ and the ξ_i's are new variables, inductively as follows:
(1) $x\varphi = \varphi_X(x)$ for $x \in X$, $c\varphi = \varphi_0(c)$ for $c \in \Sigma_0$, and
(2) $f(t_1, \ldots, t_n)\varphi = \varphi_n(f)[\xi_1 \leftarrow t_1\varphi, \ldots, \xi_n \leftarrow t_n\varphi]$ in which ξ_i is replaced with $t_i\varphi$ for any $i \leqslant n$ (cf. [23, p. 7]).

A tree homomorphism $\varphi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, Y)$ is *regular* if for every $f \in \Sigma_m$ $(m \geqslant 1)$ each $\xi_1, \ldots, \xi_m$ appears exactly once in $\varphi_m(f)$, cf. [18].

For a regular tree homomorphism $\varphi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, Y)$, the unique extension $\varphi_* : \mathrm{C}(\Sigma, X) \to \mathrm{C}(\Omega, Y)$ to contexts is obtained by setting $\varphi_*(\xi) = \xi$ (cf. [23, Proposition 10.3]). We note that the identities $(Q \cdot P)\varphi_* = Q\varphi_* \cdot P\varphi_*$ and $(t \cdot Q \cdot P)\varphi = t\varphi \cdot Q\varphi_* \cdot P\varphi_*$ hold for all $P, Q \in \mathrm{C}(\Sigma, X)$ and $t \in \mathrm{T}(\Sigma, X)$.

For a tree language $T \subseteq \mathrm{T}(\Sigma, X)$, the *syntactic morphism* and *syntactic monoid morphism* of T are, respectively, the mappings $\varphi_T : \mathcal{T}(\Sigma, X) \to \mathrm{SOA}(T)$ and $\lambda_T : \mathrm{C}(\Sigma, X) \to \mathrm{SOM}(T)$ defined by $t\varphi_T = t/\theta_T$ and $P\lambda_T = P/\mu_T$ for any $t \in \mathrm{T}(\Sigma, X)$ and $P \in \mathrm{C}(\Sigma, X)$.

Definition 5.21. A regular tree homomorphism $\varphi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, Y)$ is *full with respect to* a tree language $T \subseteq \mathrm{T}(\Omega, Y)$ if both of the mappings $\varphi\varphi_T : \mathrm{T}(\Sigma, X) \to \mathrm{SOA}(T)$ and $\varphi_*\lambda_T : \mathrm{C}(\Sigma, X) \to \mathrm{SOM}(T)$ are surjective.

An equivalent definition is:

Lemma 5.22. *A regular tree homomorphism $\varphi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, Y)$ is full with respect to $T \subseteq \mathrm{T}(\Omega, Y)$ if and only if for every $Q \in \mathrm{C}(\Omega, Y)$ and every $s \in \mathrm{T}(\Omega, Y)$ there are $P \in \mathrm{C}(\Sigma, X)$ and $t \in \mathrm{T}(\Sigma, X)$, such that, $Q\ \mu_T\ P\varphi_*$ and $s\ \theta_T\ t\varphi$.*

Lemma 5.23. *If $\varphi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, Y)$ is a regular tree homomorphism and $T \subseteq \mathrm{T}(\Omega, Y)$ then* $\mathrm{SOM}(T\varphi^{-1}) \prec \mathrm{SOM}(T)$, *and if φ is full with respect to T then* $\mathrm{SOM}(T\varphi^{-1}) \cong \mathrm{SOM}(T)$.

Proof. We note that $\varphi_* : \mathrm{C}(\Sigma, X) \to \mathrm{C}(\Omega, Y)$ is a monoid morphism. Let $S \subseteq \mathrm{C}(\Omega, Y)$ be the image of φ_*, $\precsim$ be the restriction of $\precsim_T$ to S and μ be the equivalence relation of $\precsim$. Then S/μ is a submonoid of $\mathrm{C}(\Omega, Y)/\mu_T$. We show that $P\varphi_* \precsim Q\varphi_*$ implies $P \precsim_{T\varphi^{-1}} Q$ for all $P, Q \in \mathrm{C}(\Sigma, X)$.

Suppose $P\varphi_* \precsim Q\varphi_*$ and take arbitrary $t \in \mathrm{T}(\Sigma, X)$ and $R \in \mathrm{C}(\Sigma, X)$. Then $t \cdot Q \cdot R \in T\varphi^{-1}$ implies $t\varphi \cdot Q\varphi_* \cdot R\varphi_* \in T$, what further implies $t\varphi \cdot P\varphi_* \cdot R\varphi_* \in T$, and so $t \cdot P \cdot R \in T\varphi^{-1}$, that is $P \precsim_{T\varphi^{-1}} Q$. Hence, the mapping $\psi : S/\mu \to \mathrm{C}(\Sigma, X)/\mu_{T\varphi^{-1}}$ defined by $((P\varphi_*)\mu)\psi = P\mu_{T\varphi^{-1}}$ is well-defined, order preserving and surjective. It is also a monoid morphism, since $((P\varphi_*)\mu \cdot (Q\varphi_*)\mu)\psi = ((P \cdot Q)\varphi_*\mu)\psi = (P \cdot Q)\mu_{T\varphi^{-1}} = P\mu_{T\varphi^{-1}} \cdot Q\mu_{T\varphi^{-1}} = ((P\varphi_*)\mu)\psi \cdot ((Q\varphi_*)\mu)\psi$ for all $P, Q \in \mathrm{C}(\Sigma, X)$. Hence $\mathrm{SOM}(T\varphi^{-1}) \leftarrow S/\precsim\ \subseteq \mathrm{SOM}(T)$ holds, and so $\mathrm{SOM}(T\varphi^{-1}) \prec \mathrm{SOM}(T)$.

Suppose now that φ is full with respect to T. We show the equivalence $P \precsim_{T\varphi^{-1}} Q$ iff $P\varphi_* \precsim_T Q\varphi_*$ for any $P, Q \in \mathrm{C}(\Sigma, X)$. It has already been proved that $P\varphi_* \precsim_T Q\varphi_*$ implies $P \precsim_{T\varphi^{-1}} Q$. For the converse, suppose $P \precsim_{T\varphi^{-1}} Q$ and take arbitrary $R' \in \mathrm{C}(\Omega, Y)$ and $t' \in \mathrm{T}(\Omega, Y)$. There are $R \in \mathrm{C}(\Sigma, X)$ and $t \in \mathrm{T}(\Sigma, X)$, such that $R\varphi_*\ \mu_T\ R'$ and $t\varphi\ \theta_T\ t'$. Hence, $t' \cdot Q\varphi_* \cdot R' \in T$ implies $t\varphi \cdot Q\varphi_* \cdot R\varphi_* \in T$, so $(t \cdot Q \cdot R)\varphi \in T$, i.e., $t \cdot Q \cdot R \in T\varphi^{-1}$, what further gives $t \cdot P \cdot R \in T\varphi^{-1}$. This is equivalent to $t\varphi \cdot P\varphi_* \cdot R\varphi_* \in T$, and hence $t' \cdot P\varphi_* \cdot R' \in T$, what shows that $P\varphi_* \precsim_T Q\varphi_*$. Hence $P \precsim_{T\varphi^{-1}} Q$ iff $P\varphi_* \precsim_T Q\varphi_*$, and since the mapping $\varphi_* : \mathrm{C}(\Sigma, X) \to \mathrm{C}(\Omega, Y)$ is a monoid morphism, then by Proposition 2.3, $\mathrm{SOM}(T\varphi^{-1}) \cong \mathrm{SOM}(T)$. $\square$

In the following two lemmas some connections between tree languages recognizable by a finite ordered algebra $\mathcal{A}$ and tree languages recognizable by $\mathcal{A}^\rho$ are presented. Recall that unary ranked alphabet of the algebra $\mathcal{A}^\rho$ is $\{\overline{p} \mid p \in \mathrm{Tr}(\mathcal{A})\}$; for simplicity we denote this alphabet by $\Lambda_{\mathcal{A}}$.

Suppose $\mathcal{A} = (A, \Sigma)$ is a finite algebra. Every context in $\mathrm{C}(\Sigma, A)$ corresponds to a translation in $\mathrm{Tr}(\mathcal{A})$ in a natural way: to an elementary context $f(a_1, \ldots, \xi, \ldots, a_m)$ the elementary translation $f^{\mathcal{A}}(a_1, \ldots, \xi, \ldots, a_m)$ corresponds, where $f \in \Sigma_m$ $(m > 0)$ and $a_1, \ldots, a_m \in A$. This correspondence can be extended to the mapping $-^{\mathcal{A}} : \mathrm{C}(\Sigma, A) \to \mathrm{Tr}(\mathcal{A})$ which satisfies $(P \cdot Q)^{\mathcal{A}} = P^{\mathcal{A}} \cdot Q^{\mathcal{A}}$ for all $P, Q \in \mathrm{C}(\Sigma, A)$, and $\xi^{\mathcal{A}} = 1_A$ where 1_A is the identity translation. We note that for any translation $p \in \mathrm{Tr}(\mathcal{A})$, there is a $P \in \mathrm{C}(\Sigma, A)$, such that $P^{\mathcal{A}} = p$ and this P may not be unique. In other words, $-^{\mathcal{A}}$ is a non-injective monoid epimorphism.

We also note that the mapping $-^{\mathcal{A}} : \mathrm{C}(\Sigma, A)\backslash\{\xi\} \to \mathrm{TrS}(\mathcal{A})$ is a semigroup epimorphism that assigns non-unit contexts of $\mathrm{C}(\Sigma, A)$ to non-trivial translations of $\mathcal{A}$.

Lemma 5.24. *Let $\mathcal{A} = (A, \Sigma, \leqslant)$ be a finite ordered algebra and X be a leaf alphabet disjoint from A. For any tree language $L \subseteq \mathrm{T}(\Lambda_{\mathcal{A}}, X)$ recognized by $\mathcal{A}^\rho$ there exists a regular tree homomorphism $\varphi : \mathrm{T}(\Lambda_{\mathcal{A}}, X) \to \mathrm{T}(\Sigma, X \cup A)$ and a tree language $T \subseteq \mathrm{T}(\Sigma, X \cup A)$, such that $L = T\varphi^{-1}$ and T can be recognized by a finite power $\mathcal{A}^n$ where $n = |A|$.*

Proof. Let $\alpha : X \to \mathrm{Tr}(\mathcal{A})$ be an initial assignment for $\mathcal{A}^\rho$ and $F \subseteq \mathrm{Tr}(\mathcal{A})$ be an ideal of $\mathrm{OTr}(\mathcal{A})$ such that $L = \{t \in \mathrm{T}(\Lambda_{\mathcal{A}}, X) \mid t\alpha^{\mathcal{A}^\rho} \in F\}$. Define the tree homomorphism $\varphi : \mathrm{T}(\Lambda_{\mathcal{A}}, X) \to \mathrm{T}(\Sigma, X \cup A)$ by $\varphi_X(x) = x$ for $x \in X$, and for every $p \in \mathrm{Tr}(\mathcal{A})$ choose a $\varphi_1(\overline{p}) \in \mathrm{C}(\Sigma, A)$ such that $\varphi_1(\overline{p})^{\mathcal{A}} = p$. Obviously φ is a regular tree homomorphism. Suppose that $A = \{a_1, \ldots, a_n\}$. Let F' be the ideal of $\mathcal{A}^n$ generated by $\{(p(a_1), \ldots, p(a_n)) \in A^n \mid p \in F\}$, i.e., $(b_1, \ldots, b_m) \in F'$ iff there is a $p \in F$, such that $b_j \leqslant p(a_j)$ for every $j \leqslant n$. Define the initial assignment $\beta : X \cup A \to A^n$ for $\mathcal{A}^n$ by $a\beta = (a, \ldots, a) \in A^n$ and $x\beta = \big((x\alpha)(a_1), \ldots, (x\alpha)(a_n)\big)$ for all $a \in A$ and $x \in X$. Let the tree language T be the subset of $\mathrm{T}(\Sigma, X \cup A)$ recognized by $(\mathcal{A}^n, \beta, F')$, that is $T = \{t \in \mathrm{T}(\Sigma, X \cup A) \mid t\beta^{\mathcal{A}^n} \in F'\}$.

We are proving that $L = T\varphi^{-1}$. Every tree w in $\mathrm{T}(\Lambda_{\mathcal{A}}, X)$ is of the form $w = \overline{p_1}(\overline{p_2}(\ldots\overline{p_k}(x)\ldots))$ for some $p_1, \ldots, p_k \in \mathrm{Tr}(\mathcal{A})$ $(k \geqslant 0)$ and $x \in X$. For such a tree w, $w\alpha^{\mathcal{A}^\varrho} = x\alpha \cdot p_k \cdots p_2 \cdot p_1$ and $(w\varphi)\beta^{\mathcal{A}^n} = (x\alpha \cdot p_k \cdots p_2 \cdot p_1(a_1), \ldots, x\alpha \cdot p_k \cdots p_2 \cdot p_1(a_n))$. Hence, $w\varphi \in T$ iff $(w\varphi)\beta^{\mathcal{A}^n} \in F'$, i.e., there is a $p \in F$, such that $x\alpha \cdot p_k \cdots p_2 \cdot p_1(a) \leqslant p(a)$ for every $a \in A$, or, equivalently, $x\alpha \cdot p_k \cdots p_2 \cdot p_1 \lesssim_{\mathcal{A}} p$ for some $p \in F$, what means $x\alpha \cdot p_k \cdots p_2 \cdot p_1 \in F$, i.e., $w\alpha^{\mathcal{A}^\varrho} \in F$, or equivalently $w \in L$. $\square$

Lemma 5.25. *Let $\mathcal{A} = (A, \Sigma, \leqslant)$ be a finite ordered algebra and X be a leaf alphabet disjoint from $A \cup \Sigma$. For any tree language $T \subseteq \mathrm{T}(\Sigma, X)$ recognized by $\mathcal{A}$ there exists a unary ranked alphabet Λ and a regular tree homomorphism $\varphi : \mathrm{T}(\Lambda, X \cup \Sigma_0) \to \mathrm{T}(\Sigma, X)$, such that φ is full with respect to T, and for every $z \in X \cup \Sigma_0$, $T\varphi^{-1} \cap \mathrm{T}(\Lambda, \{z\})$ can be recognized as a subset of $\mathrm{T}(\Lambda, \{z\})$ by $\mathcal{A}^\rho$.*

Proof. Let $\mathcal{B} = (B, \Sigma, \leqslant')$ be the syntactic ordered algebra of T. Then $\mathcal{B} \prec \mathcal{A}$. Suppose $T = \{t \in \mathrm{T}(\Sigma, X) \mid t\beta^{\mathcal{B}} \in F\}$, where $\beta : X \to B$ is an initial assignment for $\mathcal{B}$ and $F \trianglelefteq \mathcal{B}$.

Since $\mathcal{B}$ is the least ordered algebra that recognizes T, the algebra $\mathcal{B}$ is generated by $\beta(X)$. The mapping $\beta : X \to B$ can be uniquely extended to a monoid morphism $\beta_c : \mathrm{C}(\Sigma, X) \to \mathrm{C}(\Sigma, B)$. Since B is generated by $\beta(X)$, the mapping $\beta_c^{\mathcal{B}} : \mathrm{C}(\Sigma, X) \to \mathrm{Tr}(\mathcal{B})$, $\beta_c^{\mathcal{B}}(Q) = \beta_c(Q)^{\mathcal{B}}$ is surjective. Define the tree homomorphism $\varphi : \mathrm{T}(\Lambda_{\mathcal{B}}, X \cup \Sigma_0) \to \mathrm{T}(\Sigma, X)$ by $\varphi_X(x) = x$ for any $x \in X \cup \Sigma_0$, and for every $q \in \mathrm{Tr}(\mathcal{B})$ choose a $\varphi_1(\overline{q}) = Q \in \mathrm{C}(\Sigma, X)$, such that $\beta_c(Q)^{\mathcal{B}} = q$. Note that φ is a regular tree homomorphism. It remains to show that φ is full with respect to T and that for every $z \in X \cup \Sigma_0$, $L_z = T\varphi^{-1} \cap \mathrm{T}(\Lambda, \{z\})$ can be recognized as a subset of $\mathrm{T}(\Lambda, \{z\})$ by $\mathcal{B}^\rho$. This will finish the proof since $\mathrm{OTr}(\mathcal{B}) \prec \mathrm{OTr}(\mathcal{A})$ follows from $\mathcal{B} \prec \mathcal{A}$ by Proposition 5.2, and so $\mathcal{B}^\rho \prec \mathcal{A}^\rho$ by Proposition 5.7, which implies that L_z can also be recognized by $\mathcal{A}^\rho$.

First, we show that φ is full with respect to T. Let $Q \in \mathrm{C}(\Sigma, X)$ be a context. For $q = \beta_c(Q)^{\mathcal{B}} \in \mathrm{Tr}(\mathcal{B})$, $\overline{q}(\xi)\varphi_* \, \mu_T \, Q$ holds. By induction on the height of t we show that for any $t \in \mathrm{T}(\Sigma, X)$ there is an $s \in \mathrm{T}(\Lambda_{\mathcal{B}}, X \cup \Sigma_0)$, such that $t \, \theta_T \, s\varphi$. If $t = x \in X \cup \Sigma_0$ then $s\varphi \, \theta_T \, t$ for $s = t$. If $t = t' \cdot P$ for some $P \in \mathrm{C}(\Sigma, X)$ and $t' \in \mathrm{T}(\Sigma, X)$, such that the height of t' is less than the height of t, then, by the induction hypothesis, there is an $s' \in \mathrm{T}(\Lambda_{\mathcal{B}}, X \cup \Sigma_0)$, such that $t' \, \theta_T \, s'\varphi$. Also, $\overline{p}(\xi)\varphi_* \, \mu_T \, P$ for some $p \in \mathrm{Tr}(\mathcal{B})$ holds. Let $s = \overline{p}(s')$. Then $s\varphi = s'\varphi \cdot \overline{p}(\xi)\varphi_* \, \theta_T \, t' \cdot P = t$. The claim follows from Lemma 5.22.

Second, we are proving that L_z can be recognized by $\mathcal{B}^\rho$ for a fixed $z \in X \cup \Sigma_0$. Let 1_B be the identity translation of $\mathcal{B}$. Define the initial assignment $\alpha : \{z\} \to \mathrm{Tr}(\mathcal{B})$ for $\mathcal{B}^\rho$ by $z\alpha = 1_B$, and let $F_z = \{q \in \mathrm{Tr}(\mathcal{B}) \mid q(z\beta^{\mathcal{B}}) \in F\}$. We show that $F_z \trianglelefteq \mathcal{B}^\rho$ and L_z is recognized by $(\mathcal{B}^\rho, \alpha, F_z)$. For $p, q \in \mathrm{Tr}(\mathcal{B})$, if $p \lesssim_{\mathcal{B}} q \in F_z$ then $p(z\beta^{\mathcal{B}}) \leqslant' q(z\beta^{\mathcal{B}}) \in F$, so $p(z\beta^{\mathcal{B}}) \in F$, and hence $p \in F_z$. Thus $F_z \trianglelefteq \mathcal{B}^\rho$. Every $w \in \mathrm{T}(\Lambda_{\mathcal{B}}, \{z\})$ can be written in the form $w = \overline{q_1}(\overline{q_2}(\ldots \overline{q_h}(z) \ldots))$ for some $q_1, \ldots, q_h \in \mathrm{Tr}(\mathcal{B})$ $(h \geqslant 0)$. For such a tree w, $w\alpha^{\mathcal{B}^\rho} = 1_B \cdot q_h \cdots q_2 \cdot q_1$ and $(w\varphi)\beta^{\mathcal{B}} = q_h \cdots q_2 \cdot q_1(z\beta^{\mathcal{B}})$. Thus, $w \in L_z$ iff $w\varphi \in T$, i.e., $(w\varphi)\beta^{\mathcal{B}} \in F$, what means $q_h \cdots q_2 \cdot q_1(z\beta^{\mathcal{B}}) \in F$. This is equivalent to $q_h \cdots q_2 \cdot q_1 \in F_z$, that is $w\alpha^{\mathcal{B}^\rho} \in F_z$. Hence, $L_z = \{w \in \mathrm{T}(\Lambda, \{z\}) \mid w\alpha^{\mathcal{B}^\rho} \in F_z\}$. □

Before characterizing gPVTL's definable by syntactic ordered monoids, we note a remark.

Remark 5.26. Let Λ be a unary ranked alphabet. For every leaf alphabet X and every subset $Y \subseteq X$, $\mathrm{C}(\Lambda, Y) = \mathrm{C}(\Lambda, X)$, and the quasi-order $\precsim_T$ for a tree language $T \subseteq \mathrm{T}(\Lambda, Y)$ on $\mathrm{C}(\Lambda, Y)$ is the same relation $\precsim_T$ on $\mathrm{C}(\Lambda, X)$ when T is viewed as a subset of $\mathrm{T}(\Lambda, X)$. Therefore, if a family of tree languages $\mathscr{V} = \{\mathscr{V}(\Sigma, X)\}$ is definable by syntactic ordered monoids, then for any unary ranked alphabet Λ and any leaf alphabets X and Y, if $Y \subseteq X$ then $\mathscr{V}(\Lambda, Y) \subseteq \mathscr{V}(\Lambda, X)$.

Proposition 5.27. *A family of recognizable tree languages $\mathscr{V}$ is definable by syntactic ordered monoids if and only if $\mathscr{V}$ is a gPVTL that satisfies the following properties*:
(1) *the family $\mathscr{V}$ is closed under inverse regular tree homomorphisms*;
(2) *for every unary ranked alphabet Λ, and any leaf alphabets X and Y, if $Y \subseteq X$ then $\mathscr{V}(\Lambda, Y) \subseteq \mathscr{V}(\Lambda, X)$*;
(3) *for a regular tree homomorphism $\varphi : \mathrm{T}(\Sigma, X) \to \mathrm{T}(\Omega, Y)$ full with respect to a tree language $T \subseteq \mathrm{T}(\Omega, Y)$, if $T\varphi^{-1} \in \mathscr{V}(\Sigma, X)$ then $T \in \mathscr{V}(\Omega, Y)$.*

Proof. The fact that for any VFOM $\mathbf{M}$, $\mathbf{M}^{\mathrm{t}}$ is a gPVTL follows from Corollary 5.16, that it satisfies the conditions (1) and (3) follows from Proposition 5.23 and that it satisfies the condition (2) follows from Remark 5.26.

For the converse, suppose that a gPVTL $\mathscr{V} = \{\mathscr{V}(\Sigma, X)\}$ satisfies the conditions of the proposition. By Corollary 5.19 it is enough to show that $\mathscr{V}^{\mathrm{a}}$ satisfies the condition of Proposition 5.10.

Let $\mathcal{A} = (A, \Sigma, \leqslant)$ be a finite ordered algebra in $\mathscr{V}^{\mathrm{a}}$. By Lemma 5.24, any tree language $L \subseteq \mathrm{T}(\Lambda_{\mathcal{A}}, X)$ recognizable by $\mathcal{A}^{\rho}$ can be written as $L = T\varphi^{-1}$, where $\varphi : \mathrm{T}(\Lambda_{\mathcal{A}}, X) \to \mathrm{T}(\Sigma, X \cup A)$ is a regular tree homomorphism and T is a tree language recognized by some power $\mathcal{A}^n$ of $\mathcal{A}$. Then $\mathcal{A}^n \in \mathscr{V}^{\mathrm{a}}$ implies that $T \in \mathscr{V}(\Sigma, X \cup A)$, and hence $L = T\varphi^{-1} \in \mathscr{V}(\Lambda_{\mathcal{A}}, X)$ by (1). This holds for every tree language L recognizable by $\mathcal{A}^{\rho}$, so $\mathcal{A}^{\rho} \in \mathscr{V}^{\mathrm{a}}$ by Corollary 3.9(2).

Suppose now that $\mathcal{A}^{\rho} \in \mathscr{V}^{\mathrm{a}}$ for a finite ordered algebra $\mathcal{A} = (A, \Sigma, \leqslant)$. Let $T \subseteq \mathrm{T}(\Sigma, X)$ be a tree language recognizable by $\mathcal{A}$. By Lemma 5.25, there are a unary ranked alphabet Λ and a regular tree homomorphism $\varphi : \mathrm{T}(\Lambda, X \cup \Sigma_0) \to \mathrm{T}(\Sigma, X)$ full with respect to T, such that for every $z \in X \cup \Sigma_0$, $L_z = T\varphi^{-1} \cap \mathrm{T}(\Lambda, \{z\})$ can be recognized as a subset of $\mathrm{T}(\Lambda, \{z\})$ by $\mathcal{A}^{\rho}$. So, $L_z \in \mathscr{V}(\Lambda, \{z\})$, thus $L_z \in \mathscr{V}(\Lambda, X \cup \Sigma_0)$ by (2). Hence, $T\varphi^{-1} = \bigcup_{z \in X \cup \Sigma_0} L_z \in \mathscr{V}(\Lambda, X \cup \Sigma_0)$. Since φ is full with respect to T, then $T \in \mathscr{V}(\Sigma, X)$ by (3). This holds for every tree language T recognizable by $\mathcal{A}$, so $\mathcal{A} \in \mathscr{V}^{\mathrm{a}}$ by Corollary 3.9(2). □

5.4. Examples

Corollary 5.19, Proposition 4.5 and conclusions from Section 5.2 imply that gPVTL Cof is not definable by syntactic ordered monoids, family SL is definable by syntactic monoids, also family Sym is definable by syntactic ordered monoids. Anyway, these can be verified directly.

Let $\Lambda = \Lambda_1 = \{f\}$ be a unary ranked alphabet and $X = \{x, y\}$, $Y = \{y\}$ be leaf alphabets. The language $T_1 = \{f(f(x)), f(f(f(x))), \ldots\}$ is not cofinite in $T(\Lambda, X)$, whereas the language $T_2 = \{f(f(y)), f(f(f(y))), \ldots\}$ is cofinite in $T(\Lambda, Y)$. However, they have isomorphic syntactic ordered monoids. Therefore, Cof is not definable by syntactic ordered monoids. The same conclusion follows from Proposition 5.27, since $T_2 \in \mathrm{Cof}(\Lambda, \{y\})$, but $T_2 \notin \mathrm{Cof}(\Lambda, X)$, and hence Cof does not satisfy condition (2) of the proposition.

Family SL is definable by syntactic monoids, since a tree language is semilattice if and only if its translation monoid is a semilattice monoid.

A tree language is symbolic if and only if its ordered translation monoid is a symbolic ordered monoid, thus family Sym is definable by syntactic ordered monoids.

6. Conclusions

A variety theorem connecting families of recognizable tree languages to classes of finite ordered algebras and a generalized form of the above variety theorem have been proved in the paper. Besides that, classes of finite ordered algebras, as well as families of recognizable tree languages, definable by ordered monoids have been characterized. Three examples have been studied along the paper:

(1) family Cof of cofinite tree languages, which is a gPVTL, characterizable by ordered nilpotent algebras, but not definable by ordered monoids or semigroups,
(2) family SL of semilattice tree languages, which is a generalized variety of tree languages, characterizable by semilattice algebras and definable by semilattice monoids, and
(3) family Sym of symbolic tree languages, which is a gPVTL, characterizable by symbolic ordered algebras and definable by symbolic ordered monoids.

7. Index of notation

Notation	Explanation	Page
$\preccurlyeq, \precsim$	Quasi-orders	3
$\leqslant, \lesssim$	Orders	3
$\subseteq, \subseteq_g$	Order (g-)subalgebra, Subset	4, 20
$\leftarrow, \leftarrow_g$	Order (g-)epimorphic image	4, 20
$\prec, \prec_g$	(g-)divides	4, 20
$\cong, \cong_g$	Order (g-)isomorphism	4, 20
$\mathcal{A} \times \mathcal{B}, \kappa(\mathcal{A}_1, \ldots, \mathcal{A}_n)$	Direct (g-)product	4, 21
(g)VFOA	(g-)Variety of finite ordered algebras	4, 21
$\mathcal{A}/\preccurlyeq$	Quotient ordered algebra	4
$\varphi \circ \preccurlyeq \circ \varphi^{-1}$	Inverse image of $\preccurlyeq$ under φ	5
$\mathrm{Tr}(\mathcal{A})$	Translation monoid of the algebra $\mathcal{A}$	5
$I \trianglelefteq \mathcal{A}$	Ideal	5
$\preccurlyeq_I, \theta_I$	Syntactic quasi-order and congruence of I	5, 6
$\mathrm{TrS}(\mathcal{A})$	Translation semigroup of $\mathcal{A}$	7
$\mathbf{Nil}(\Sigma)$	Variety of ordered nilpotent Σ-algebras	7
$\mathbf{SL}(\Sigma)$	Variety of semilattice Σ-algebras	9
$\mathbf{Sym}(\Sigma)$	Variety of symbolic ordered Σ-algebras	9
$\mathrm{T}(\Sigma, X), \mathrm{C}(X, T)$	Set of ΣX-trees and ΣX-contexts	10, 10
$\preccurlyeq_T, \theta_T$	Syntactic quasi-order and congruence of T	10, 10
$\mathrm{SOA}(T)$	Syntactic ordered algebra of T	10
$(\mathcal{A}, \alpha, I)$	Tree recognizer	11
$\alpha^{\mathcal{A}}$	Extension of an initial assignment α for $\mathcal{A}$	11
(g)PVTL	Positive (g-)variety of tree languages	12, 21
$\mathscr{K}^t, \mathscr{V}^a$	Variety operations	12, 21
$\mathrm{Cof}(\Sigma, X), \mathrm{Cof}_\Sigma$	Cofinite tree languages	14
$\mathrm{c}(t)$	Contents of tree t	14
$\mathrm{Sym}(\Sigma, X), \mathrm{Sym}_\Sigma$	Symbolic tree languages	15
$\mathrm{SL}(\Sigma, X), \mathrm{SL}_\Sigma$	Semilattice tree languages	15
$\mathrm{OTr}(\mathcal{A}) = (\mathrm{Tr}(\mathcal{A}), \cdot, \lesssim_{\mathcal{A}})$	Ordered translation monoid of $\mathcal{A}$	23
VFOM	Variety of finite ordered monoids	24
$\mathbf{M}^{\mathrm{a}}, \mathbf{M}^{\mathrm{t}}$	Variety operations on VFOM $\mathbf{M}$	4, 28
$\Lambda_D = \{\overline{d} \mid d \in D\}$	Unary ranked alphabet associated with D	25
$\mathcal{M}^{\nu} = (M, \Lambda_M, \lesssim)$	Unary ranked algebra associated with $\mathcal{M}$	25

$\mathcal{A}^\rho$	Unary algebra associated with $\mathcal{A}$	25
$\precsim_T$	Quasi-order on contexts	27
μ_T	Syntactic m-congruence of T	27
$\mathrm{SOM}(T)$	Syntactic ordered monoid of T	27

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References

[1] J. Almeida, On pseudovarieties, varieties of languages, filters of congruences, pseudoidentities and related topics, Algebra Universalis 27 (1990) 333–350.
[3] S.L. Bloom, Varieties of ordered algebras, J. Comput. System Sci. 13 (1976) 200–212.
[4] S.L. Bloom, B.J. Wright, P-varieties: a signature independent characterization of varieties of ordered algebras, J. Pure Appl. Algebra 29 (1983) 13–58.
[5] S. Eilenberg, Automata, Languages, and Machines, Vol. B, Pure and Applied Mathematics, Vol. 59, Academic Press, New York, London, 1976.
[6] Z. Ésik, A variety theorem for trees and theories, Automata and formal languages VIII (Salgótarján, 1996), Publ. Math. Debrecen 54 (1999) 711–762.
[7] Z. Ésik, P. Weil, On logically defined recognizable tree languages, in: Proc. FSTTCS'03, Lecture Notes in Computer Science, Vol. 2914, Springer, Berlin, 2003, pp. 195–207.
[8] A.C. Gómez, J.E. Pin, Shuffle on positive varieties of languages, Theoret. Comput. Sci. 312 (2004) 433–461.
[9] N. Kehayopulu, M. Tsingelis, Pseudoorder in ordered semigroups, Semigroup Forum 50 (1995) 389–392.
[10] M. Nivat, A. Podelski, Tree monoids and recognizability of sets of finite trees, in: H. Aït-Kaci, M. Nivat (Eds.), Resolution of Equations in Algebraic Structures, Vol. 1, Academic Press, Boston, MA, 1989, pp. 351–367.
[11] T. Petković, S. Salehi, Positive varieties of tree languages, TUCS Technical Reports 622, September 2004. URL: http://www.tucs.fi/publications/insight.php?id = tSaPe04a.
[12] J.E. Pin, Varieties of formal languages, in: Foundations of Computer Science, Plenum Publishing, New York, 1986.
[13] J.E. Pin, A variety theorem without complementation, Izvestiya VUZ Mat. 39 (1995) 80–90 (English version, Russian Mathem. (Iz. VUZ) 39 (1995) 74–63).
[14] J.E. Pin, Positive varieties and infinite words, in: C.L. Lucchesi, A.V. Moura (Eds.), LATIN'98: Theoretical Informatics, Lecture Notes in Computer Science, Vol. 1380, Springer, Berlin, 1998, pp. 76–87.
[15] A. Podelski, A monoid approach to tree languages, in: M. Nivat, A. Podelski (Eds.), Tree Automata and Languages, Elsevier, Amsterdam, 1992, pp. 41–56.
[16] S. Salehi, Varieties of tree languages definable by syntactic monoids, Acta Cybernet. 17 (2005) 21–41.
[17] S. Salehi, Varieties of tree languages, Ph.D. Thesis, Department of Mathematics, University of Turku, TUCS Dissertations 64, 2005.
[18] K. Salomaa, Syntactic monoids of regular forests, M.Sc. Thesis, Department of Mathematics, Turku University, 1983 (in Finnish).
[19] M.P. Schützenberger, On finite monoids having only trivial subgroups, Inform. Control 8 (1965) 190–194.
[20] D. Scott, The lattice of flow diagrams, in: 1971 Symp. on Semantics of Algorithmic Languages, Lecture Notes in Mathematics, Vol. 188, Springer, Berlin, pp. 311–366.
[21] M. Steinby, Syntactic algebras and varieties of recognizable sets, in: Proceedings CAAP'79, University of Lille, 1979, pp. 226–240.
[22] M. Steinby, A theory of tree language varieties, in: M. Nivat, A. Podelski (Eds.), Tree Automata and Languages, Elsevier, Amsterdam, 1992, pp. 57–81.

[23] M. Steinby, General varieties of tree languages, Theoret. Comput. Sci. 205 (1998) 1–43.
[24] D. Thérien, Recognizable languages and congruences, Semigroup Forum 23 (1981) 371–373.
[25] W. Thomas, Logical aspects in the study of tree languages, in: 9th Colloq. on Trees in Algebra and in Programming (Proc. CAAP'84), Cambridge University Press, Cambridge, 1984, pp. 31–51.
[26] W. Wechler, Universal algebra for computer scientists, in: EATCS Monographs on Theoretical Computer Science, Vol. 25, Springer, Berlin, 1992.
[27] T. Wilke, An algebraic characterization of frontier testable tree languages, Theoret. Comput. Sci. 154 (1996) 85–106.