

Saeed SALEHI

ON RUDIMENTARITY, PRIMITIVE RECURSIVITY AND REPRESENTABILITY

A b s t r a c t. It is quite well-known from Kurt Gödel's (1931) ground-breaking Incompleteness Theorem that rudimentary relations (i.e., those definable by bounded formulae) are primitive recursive, and that primitive recursive functions are representable in sufficiently strong arithmetical theories. It is also known, though perhaps not as well-known as the former one, that some primitive recursive relations are not rudimentary. We present a simple and elementary proof of this fact in the first part of the paper. In the second part, we review some possible notions of representability of functions studied in the literature, and give a new proof of the equivalence of the weak representability with the (strong) representability of functions in sufficiently strong arithmetical theories.

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1. Introduction and Preliminaries

Primitive recursive functions are what were called “rekursiv” by Kurt Gödel in his seminal 1931 paper [5] where he proved the celebrated incompleteness theorem. The main features of the primitive recursive functions used by Gödel were the following:

1. They are *computable* (i.e., for each primitive recursive function there exists an algorithm that computes it). However, we now know that they do not make up the whole (intuitively) computable functions (from tuples of natural numbers to natural numbers, $\mathbb{N}^k \rightarrow \mathbb{N}$). So, “rekursiv” functions are now called “primitive recursive” functions, which constitute a sub-class of *recursive functions* that, by Alonzo Church’s Thesis, are believed to constitute the whole computable functions.

2. They are *representable* in (sufficiently expressive and sufficiently strong) formal arithmetical theories. It is now known that, more generally, (only) recursive functions are representable in all the recursively enumerable, sufficiently strong and sufficiently expressive theories (see Section 3).

3. Theories whose *set of axioms* are primitive recursive and extend a base theory (such as Robinson’s Arithmetic \mathcal{Q}), are incomplete. It was later found out that this holds more generally for recursively enumerable extensions of \mathcal{Q} . Also, by William Craig’s Trick, every such theory is equivalent with another theory whose set of axioms is rudimentary (i.e., definable by a bounded formula).

Even though one can set up the whole theory of computable functions (aka recursion theory) and the incompleteness theorems without introducing the notion of primitive recursive functions (and relations), the theory of primitive recursive functions is a main topic in the literature on recursive function theory and the incompleteness theorems. For the sake of completeness we review some basic notions of this theory.

Definition 1.1 (Primitive Recursive Functions and Relations). The class of *primitive recursive* (PR) functions ($\mathbb{N}^k \rightarrow \mathbb{N}$) is the smallest class that contains the initial functions (the constant zero, the successor $\mathfrak{s}(x) = x + 1$, and the projection functions) and is closed under primitive recursion and composition of functions. A relation $R \subseteq \mathbb{N}^k$ is called PR, if its characteristic function ($\chi_R(\vec{n}) = 1$ if $\vec{n} \in R$, and $\chi_R(\vec{n}) = 0$ if $\vec{n} \notin R$) is PR.

By Hermann Grassmann’s recursive definition of the addition and multiplication, i.e. $x+0=x$, $x+\mathfrak{s}(y)=\mathfrak{s}(x+y)$, $x\cdot 0=0$ and $x\cdot\mathfrak{s}(y)=(x\cdot y)+x$, it can be shown that these functions are PR; so is the anti-sign function ($\tilde{\mathfrak{s}}\mathfrak{g}(0)=1$, and $\tilde{\mathfrak{s}}\mathfrak{g}(x)=0$ if $x>0$). The equality ($=$) and inequality (\leq) can be shown to be PR relations. The following identities show that the class of PR relations is closed under Boolean operations and bounded quantifications:

$$\begin{aligned} \chi_{R^c} &= \tilde{\mathfrak{s}}\mathfrak{g}(\chi_R); & \chi_{R\cap S} &= \chi_R \cdot \chi_S. \\ \left\{ \begin{aligned} \chi_{\forall x \leq y R(\vec{z}, x)}(\vec{z}, 0) &= \chi_R(\vec{z}, 0), \\ \chi_{\forall x \leq y R(\vec{z}, x)}(\vec{z}, \mathfrak{s}(y)) &= \chi_{\forall x \leq y R(\vec{z}, x)}(\vec{z}, y) \cdot \chi_R(\vec{z}, y+1). \end{aligned} \right. \end{aligned}$$

Definition 1.2 (Rudimentary Relations). A formula in the language of arithmetic $\langle 0, 1, +, \times, \leq \rangle$ is called *bounded*, if it has been constructed from atomic formulas (of the form $u = v$ or $u \leq v$, for terms u, v) by means of Boolean connectives, and bounded quantifications (of the form e.g. $\forall x \leq t$, where $\forall x \leq t \mathcal{A}(x, t)$ abbreviates the formula $\forall x [x \leq t \rightarrow \mathcal{A}(x, t)]$ for term t and variable x which is not free in t). The class of bounded formulas is denoted by Δ_0 . A relation $R \subseteq \mathbb{N}^k$ is called *rudimentary* or *bounded definable*, or simply Δ_0 , if it can be defined by a Δ_0 -formula, i.e., there exists a Δ_0 -formula $\varphi(\vec{x})$ such that $R = \{\vec{m} \mid \mathbb{N} \models \varphi(\vec{m})\}$.

Thus, all the Δ_0 relations are PR; see also e.g. [2, 7, 17]. The question as to whether the converse holds, i.e., *whether every PR relation is Δ_0* , has been mentioned in very few places; some of which, unfortunately, are wrong (cf. e.g. [7, Exercise 8.6]) or misleading (cf. e.g. [17, Section 6.3, Remark 1])—see [18] for more details. It may seem that the graphs of (very) fast-growing functions could be non-rudimentary, but, in fact, it has been shown in [1] (see also [4]) that this is not true.

We read in the Abstract of [4], “The question of whether a given primitive recursive relation is rudimentary is in some cases difficult and related to several well-known open questions in theoretical computer science”. Also, on page 130 of [4] we read, “However, it is difficult to exhibit a *natural* arithmetical relation which can be proved not to be rudimentary” (emphasis in the original). Later, on page 132 we read, “Hence, the main way of exhibiting a primitive recursive relation which is not rudimentary is to choose it in $\mathfrak{C}_*^3 \setminus \mathfrak{C}_*^2$. Although it is true that infinitely many [such] relations exist, we know no natural example”. Here, by “*natural*” the authors mean a relation ($\subseteq \mathbb{N}^k$) that the number-theorists use and work with.

On page 85 of [2] after proving that “Every Δ_0 relation is primitive recursive” as a **Lemma**, we read, “**Remark:** The converse of the above lemma is false, as can be shown by a diagonal argument. For those familiar with complexity theory, we can clarify things as follows. As noted in the Side Remark above, all Δ_0 relations can be recognized in linear space on a Turing machine. On the other hand, it follows from the Ritchie-Cobham Theorem that all relations recognizable in space bounded by a primitive recursive function of the input length are primitive recursive. In particular, space $O(n^2)$ relations are primitive recursive, and a straightforward diagonal argument shows that there are relations recognizable in n^2 space which are not recognizable in linear space, and hence are not Δ_0 relations.” The mentioned side-remark (that “All Δ_0 relations can be recognized in linear space on a Turing machine, when input numbers are represented in binary notation”) is not proved in [2]. This was proved first by John Myhill [14].

So, there are some PR relations that are not Δ_0 . In Section 2 we show that a specific PR relation is not Δ_0 , by a detailed proof with little background in complexity theory or formal arithmetics. This relation may not look natural for number-theorists, but is sufficiently natural for logicians.

Remark 1.3 (The Bounds on Quantifiers in Defining Formulas). If a relation is defined by a formula whose all quantifiers are bounded by polynomials, then that relation is Δ_0 (and thus PR). If the quantifiers of such a formula are bounded by PR functions (which are not necessarily polynomials), then that relation is surely PR (recall that the PR functions are closed under substitutions); but it may not be Δ_0 , as will be clear below (see the proof of Theorem 2.8 and the defining formula of $\text{Sat}_{\Delta_0}(x, y)$ which is a non-rudimentary PR relation).

In the second part, Section 3, we will study some possible notions of representability of functions in arithmetical theories and will compare their strength with each other; we will provide a new proof for an old theorem which appears in a very few places with a much longer proof. The theorem says that *every weakly representable function is (strongly) representable*; this is usually proved by showing that (A) every weakly representable function is recursive, and (B) every recursive function is (strongly) representable. Our proof is direct and much more elementary.

2. Rudimentarity vs. Primitive Recursivity

Let us be given a fixed Gödel coding $\alpha \mapsto \ulcorner \alpha \urcorner$, which is primitive recursive (as is usually presented in the literature). Our example of a PR relation that is not Δ_0 , uses an idea of Alfred Tarski; that the truth relation of arithmetical sentences is not arithmetically definable. Likewise, the truth of Δ_0 -sentences is not Δ_0 ; but, as will be shown later, it is PR.

Definition 2.1 (Δ_0 -Satisfaction). Let \mathbf{Sat}_{Δ_0} be the set of all the ordered pairs $(\ulcorner \theta(\vec{v}) \urcorner, a)$, where $\theta(\vec{v})$ is a Δ_0 -formula with the shown free variables and $a \in \mathbb{N}$, such that $\mathbb{N} \models \theta(\vec{a})$; i.e., the sentence resulted from substituting a for every free variable of θ is true (in the standard model of natural numbers).

In the other words, $\mathbf{Sat}_{\Delta_0} = \{(\ulcorner \theta(\vec{v}) \urcorner, a) \mid \mathbb{N} \models \theta(\vec{a}) \ \& \ \theta \in \Delta_0\}$.

Theorem 2.2 (Non-Rudimentarity of Δ_0 -Satisfaction). *The relation $\mathbf{Sat}_{\Delta_0}(x, y)$ is not definable by any Δ_0 -formula.*

Proof. If a Δ_0 -formula such as $\sigma(x, y)$ defined \mathbf{Sat}_{Δ_0} , then for the Δ_0 -formula $\theta(x) = \neg\sigma(x, x)$ and $m = \ulcorner \theta(x) \urcorner$, we would have $\mathbb{N} \models \theta(m) \leftrightarrow \mathbf{Sat}_{\Delta_0}(\ulcorner \theta(x) \urcorner, m) \leftrightarrow \sigma(m, m) \leftrightarrow \neg\theta(m)$, a contradiction! \square

In the rest of this section we show that \mathbf{Sat}_{Δ_0} is a PR relation. This can already be inferred from the results of [12], see [12, Definition 4.1.3 and Lemma 4.1.4] and also [16, Theorem 2] and [6, Corollary 5.5]. All of these use advanced arguments that cannot be mentioned in more elementary texts like [2, 7, 17]. Our aim here is to provide an elementary proof for primitive recursivity of \mathbf{Sat}_{Δ_0} in such a way that it can be used, along with Theorem 2.2, in textbooks for clarifying the status of PR vs. Δ_0 relations.

Remark 2.3 (On Gödel Coding). We can assume that the set of the Gödel codes of the variables is definable by a Δ_0 -formula; for example we can keep even numbers $2, 4, 6, \dots$ for coding the variables v_0, v_1, v_2, \dots respectively, and then code the rest of the language (propositional connectives, quantifiers, parentheses and function and relation symbols) by odd numbers. As a result of this way of coding, $\mathbf{var}(x) \equiv \exists y \leq x (y = 2x + 2)$ is a Δ_0 -formula that defines the variables. Other syntactical notions of *terms*, *formulas*, *sentences*, *bounded sentences*, *proofs*, etc. can be shown to be PR as usual (see e.g. [6, 7, 13, 17]). Let $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \dots$ be the sequence of all

prime numbers $(2, 3, 5, \dots)$. Let us code the sequence $\langle \alpha_0, \alpha_1, \dots, \alpha_k \rangle$ by the number $\prod_{i \leq k} \mathfrak{p}_i^{\alpha_i + 1}$. Let us note that this way, the code of any such sequence will be non-greater than \mathfrak{p}_k^{kA} , where A is any number greater than all α_i 's. Also let us recall that the functions $i \mapsto \mathfrak{p}_i$ and $(k, A) \mapsto \mathfrak{p}_k^{kA}$ are both PR (see e.g. [7, 13, 17]).

Definition 2.4 (Terms, Bounded Formulas, Valuations, etc.). For a fixed Gödel coding, let the relation

- $\mathbf{var}(x)$ hold, when “ x is (the Gödel code of) a *variable*”.
- $\mathbf{trm}(x)$ hold, when “ x is (the Gödel code of) a *term*”.
- $\mathbf{atm}(x)$ hold, when “ x is (the Gödel code of) an *atomic formula*”.
- $\mathbf{fml}_{\Delta_0}(x)$ hold, when “ x is (the Gödel code of) a Δ_0 -*formula*”.
- $\mathbf{val}(x, y, z)$ hold, when “ x is (the Gödel code of) a term with the free variables $\langle \nu_0, \dots, \nu_\ell \rangle$, y is (the Gödel code of) a sequence of numbers $\langle a_0, \dots, a_\ell \rangle$, and z is the *value* of the term x when each ν_i is substituted with a_i , for each $i \leq \ell$ ”.

Lemma 2.5 (\mathbf{var} , \mathbf{trm} , \mathbf{fml}_{Δ_0} and \mathbf{val} are PR). *The relations \mathbf{var} , \mathbf{trm} , \mathbf{atm} , \mathbf{fml}_{Δ_0} and \mathbf{val} are PR.*

Proof. We noted (in Remark 2.3) that \mathbf{var} can even be Δ_0 (and so it is PR) by a modest convention on coding. There is also a Δ_0 relation $\mathbf{seq}(x)$ which holds of x when x is (the Gödel code of) a sequence. Let $\mathit{len}(x)$ denote the length of x and $[x]_i$, for each $i < \mathit{len}(x)$, denote the i -th element of x . Thus, if $\mathbf{seq}(x)$ holds, then x codes the sequence $\langle [x]_0, [x]_1, \dots, [x]_{\mathit{len}(x)-1} \rangle$. Let us recall that $x \mapsto \mathit{len}(x)$ and $(i, x) \mapsto [x]_i$ are both PR functions. Let $y = \mathit{last}(x)$ abbreviate $y = [x]_{\mathit{len}(x)-1}$.

- Let $\mathbf{trmseq}(x)$ be the following Δ_0 relation:

$$\mathbf{seq}(x) \wedge \forall i < \mathit{len}(x) \left[[x]_i = \ulcorner 0 \urcorner \vee [x]_i = \ulcorner 1 \urcorner \vee \right. \\ \left. \mathbf{var}([x]_i) \vee \exists j, k < i \left([x]_i = \ulcorner ([x]_j + [x]_k) \urcorner \vee [x]_i = \ulcorner ([x]_j \times [x]_k) \urcorner \right) \right].$$

Now, $\mathbf{trm}(x)$ can be written as $\exists s \leq \mathfrak{p}_x^{(x+1)^2} \mathbf{trmseq}(s) \wedge \mathit{last}(s) = x$; noting that the length of the building sequence of x is at most x and all the elements of that sequence are non-greater than x . So, $\mathbf{trm}(x)$ is PR.

- That $\mathbf{atm}(x)$ is a PR relation, follows from the following:

$$\mathbf{atm}(x) \equiv \exists u, v < x \left[\mathbf{trm}(u) \wedge \mathbf{trm}(v) \wedge (x = \ulcorner (u = v) \urcorner \vee x = \ulcorner (u \leq v) \urcorner) \right].$$

- Without loss of generality we may assume that the propositional connectives are only \neg and \wedge and the only quantifier is \forall . Now, the following Δ_0 -formula defines the building sequence of a bounded formula:

$$\begin{aligned} \text{fml}_{\Delta_0} \text{seq}(x) &\equiv \\ \text{seq}(x) \wedge \forall i < \text{len}(x) &\left[\text{atm}([x]_i) \vee \exists j, k < i \left([x]_i = \ulcorner \neg [x]_j \urcorner \vee \right. \right. \\ [x]_i = \ulcorner [x]_j \wedge [x]_k \urcorner &\vee \exists v, t < x \left[\text{var}(v) \wedge \text{trm}(t) \wedge [x]_i = \ulcorner (\forall v \leq t) [x]_j \urcorner \right] \left. \right) \right]. \end{aligned}$$

So, $\text{fml}_{\Delta_0}(x) \equiv \exists s \leq \mathfrak{p}_x^{(x+1)^2} \text{fml}_{\Delta_0} \text{seq}(s) \wedge \text{last}(s) = x$ is a PR relation.

- Let $\text{valseq}(y, s, t)$ be the following Δ_0 relation:

$$\begin{aligned} \text{seq}(y) \wedge \text{termseq}(s) \wedge \text{seq}(t) \wedge \text{len}(t) = \text{len}(s) \wedge \forall i < \text{len}(s) \\ \left[([s]_i = \ulcorner 0 \urcorner \wedge [t]_i = 0) \vee ([s]_i = \ulcorner 1 \urcorner \wedge [t]_i = 1) \vee (\text{var}([s]_i) \wedge [t]_i = [y]_i) \vee \right. \\ \exists j, k < i \left[([s]_i = \ulcorner [s]_j + [s]_k \urcorner \wedge [t]_i = [t]_j + [t]_k) \right. \\ \left. \left. \vee ([s]_i = \ulcorner [s]_j \times [s]_k \urcorner \wedge [t]_i = [t]_j \cdot [t]_k) \right] \right], \end{aligned}$$

stating that y, t are (the Gödel code of) sequences (of numbers) and s is (the Gödel code of) a building sequence of a term such that t is the result of substituting the variables of s with the corresponding elements of y .

- Finally, $\text{val}(x, y, z)$ is PR since it is equivalent with

$$\exists s \leq \mathfrak{p}_x^{(x+1)^2} \exists t \leq \mathfrak{p}_z^{(z+1)^2} \text{valseq}(y, s, t) \wedge \text{last}(s) = x \wedge \text{last}(t) = z. \quad \square$$

Remark 2.6 (var , trm , fml_{Δ_0} and val Could Even Be Δ_0). Actually, by the techniques of [6, Chapter V] one can show that all the relations $\text{var}(x)$, $\text{trm}(x)$, $\text{atm}(x)$, $\text{fml}_{\Delta_0}(x)$ and $\text{val}(x, y, z)$ can be Δ_0 , under a suitable Gödel coding. The coding used in [6, Chapter V] has the property that the code of $\langle \alpha_0, \dots, \alpha_k \rangle$ is bounded by $9^k(\alpha_0 + 1)^2 \dots (\alpha_k + 1)^2$; see [6, Lemma 3.7]. This enables us to write defining Δ_0 -formulas for $\text{var}(x)$, $\text{trm}(x)$, $\text{atm}(x)$, $\text{fml}_{\Delta_0}(x)$ and $\text{val}(x, y, z)$. For $\text{var}(x)$ see Remark 2.3, and for $\text{trm}(x)$ see [6, Lemma 3.33]; by similar techniques $\text{atm}(x)$, $\text{fml}_{\Delta_0}(x)$ and $\text{val}(x, y, z)$ can also be defined by some Δ_0 -formulas.

Remark 2.7 (Sat_{Δ_0} In the Border of PR and Δ_0). The main idea of the proofs of Lemma 2.5 and Theorem 2.8 are from [9, Chapter 9].¹ In Theorem 2.8 we will show that $\text{Sat}_{\Delta_0}(x, y)$ is a PR relation, which, by Theorem 2.2, cannot be Δ_0 under any Gödel coding. We will see in the proof of Theorem 2.8 that Sat_{Δ_0} is definable by the relations var , trm , atm , fml_{Δ_0} and val . So, we have a boundary result here: the PR relations $\text{var}(x)$, $\text{trm}(x)$, $\text{atm}(x)$, $\text{fml}_{\Delta_0}(x)$ and $\text{val}(x, y, z)$ all can be Δ_0 under some

¹ As a referee of this journal remarked, Richard Kaye wrote in the beginning of [9, Chapter 9] (on page 104) that “This is the chapter that no one wanted to have to write”, and Laurence Kirby wrote in a review of [9] that “The trouble is that probably no one will want to have to read it either” (see [10, page 462]).

Gödel coding (see Remark 2.6), while the PR relation $\mathbf{Sat}_{\Delta_0}(x, y)$ can never be Δ_0 under any Gödel coding.

Theorem 2.8 (\mathbf{Sat}_{Δ_0} is **PR**). *The relation $\mathbf{Sat}_{\Delta_0}(x, y)$ is PR.*

Proof. Define the relation $\mathbf{sat}_{\Delta_0}\mathbf{seq}(s, t)$ by “ s is a building sequence of a Δ_0 -formula, and t is a sequence of triples $\langle i, z, w \rangle$ in which $i < \mathit{len}(s)$ and $w \leq 1$ is a truth value (1 for truth and 0 for falsity) of the formula $[s]_i$ when the variables v_0, v_1, \dots are interpreted by $[z]_0, [z]_1, \dots$ respectively”. Let $z[r/k]$ denote the sequence resulted from z by substituting its k -th element with r . The function $z, r, k \mapsto z[r/k]$ is PR, and when $\mathbf{val}(u, z, x)$ holds, then we can have $\mathbf{val}(u, z, x)$ for some $x \leq \mathbf{p}_u^{1+z^u}$, since the value of a term u when its free variables are substituted by the elements of z is non-greater than $\mathbf{p}_u^{1+z^u}$. The following formula defines the relation $\mathbf{sat}_{\Delta_0}\mathbf{seq}(s, t)$:

$$\begin{aligned} & \mathbf{fml}_{\Delta_0}\mathbf{seq}(s) \wedge \mathbf{seq}(t) \wedge \forall l < \mathit{len}(t) \\ & \exists i, z, w \leq t \left[[t]_l = \langle i, z, w \rangle \wedge i < \mathit{len}(s) \wedge w \leq 1 \wedge \right. \\ & \left([\exists u, v \leq s (\mathbf{trm}(u) \wedge \mathbf{trm}(v) \wedge [s]_i = \ulcorner (u=v) \urcorner \wedge \right. \\ & \quad [w = 1 \leftrightarrow \exists x \leq \mathbf{p}_{u+v}^{(1+z^u+v)^2} \mathbf{val}(u, z, x) \wedge \mathbf{val}(v, z, x)])] \vee \\ & \quad [\exists u, v \leq s (\mathbf{trm}(u) \wedge \mathbf{trm}(v) \wedge [s]_i = \ulcorner (u \leq v) \urcorner \wedge \\ & \quad [w = 1 \leftrightarrow \exists x, y \leq \mathbf{p}_{u+v}^{(1+z^u+v)^2} \mathbf{val}(u, z, x) \wedge \mathbf{val}(v, z, y) \wedge x \leq y])] \vee \\ & \quad [\exists j < i ([s]_i = \ulcorner \neg [s]_j \urcorner \wedge \exists p < l \exists w' \leq 1 ([t]_p = \langle j, z, w' \rangle \wedge \\ & \quad [w = 1 \leftrightarrow w' = 0])] \vee \\ & \quad [\exists j, k < i ([s]_i = \ulcorner ([s]_j \wedge [s]_k) \urcorner \wedge \exists p, q < l \exists w', w'' \leq 1 \\ & \quad ([t]_p = \langle j, z, w' \rangle \wedge [t]_q = \langle k, z, w'' \rangle \wedge [w = 1 \leftrightarrow w' = 1 \wedge w'' = 1])] \vee \\ & \quad [\exists j < i \exists u, v < s (\mathbf{trm}(u) \wedge \mathbf{var}(v) \wedge [s]_i = \ulcorner (\forall v \leq u) [s]_j \urcorner \wedge \\ & \quad \exists x \leq \mathbf{p}_u^{z^u+1} [\mathbf{val}(u, z, x) \wedge \\ & \quad \forall r \leq x \exists p < l \exists w' \leq 1 ([t]_p = \langle j, z[r/\ulcorner v \urcorner], w') \rangle] \wedge \\ & \quad [w = 1 \leftrightarrow \forall r \leq x \exists p < l \exists w' \leq 1 ([t]_p = \langle j, z[r/\ulcorner v \urcorner], 1 \rangle) \rangle] \vee \\ & \quad [w = 1 \leftrightarrow \forall r \leq x \exists p < l \exists w' \leq 1 ([t]_p = \langle j, z[r/\ulcorner v \urcorner], 1 \rangle) \rangle] \left. \right) \right]. \end{aligned}$$

Therefore, $\mathbf{sat}_{\Delta_0}\mathbf{seq}(s, t)$ is a PR relation, and so is $\mathbf{Sat}_{\Delta_0}(x, y)$ which can be written as

$$\begin{aligned} & \exists s \leq \mathbf{p}_x^{(x+1)^2} \exists t \leq \mathbf{p}_{x^2}^{2\mathbf{p}_x^{(x+1)^2}} \cdot 3^{\mathbf{p}_x^{(y+1)^2}} \cdot 5 \\ & \left[\mathbf{sat}_{\Delta_0}\mathbf{seq}(s, t) \wedge \mathit{last}(s) = x \wedge \mathit{last}(t) = \langle \mathit{len}(s) - 1, y, 1 \rangle \right]. \end{aligned}$$

Let us note that we took \neg, \wedge as the only propositional connectives and \forall as the only quantifier; and we coded $\langle i, z, w \rangle$ as $2^i \cdot 3^z \cdot 5^w$ which imply the desirable PR bounds as indicated. \square

3. Representability in Arithmetical Theories

For (total) functions we can have four different definitions for representability (originated from [20]) in arithmetical theories whose languages contain terms \bar{n} for indicating (each) $n \in \mathbb{N}$.

Definition 3.1 (Weakly Representable Functions). A function $f: \mathbb{N} \rightarrow \mathbb{N}$ is *weakly representable* in a theory T , if for some formula $\varphi(x, y)$ we have the following for every $n, m \in \mathbb{N}$:

- (1) if $f(n) = m$, then $T \vdash \varphi(\bar{n}, \bar{m})$; and
- (2) if $f(n) \neq m$, then $T \not\vdash \varphi(\bar{n}, \bar{m})$.

Definition 3.2 (Representable Functions). A function $f: \mathbb{N} \rightarrow \mathbb{N}$ is *representable* in a theory T , if for some formula $\psi(x, y)$ we have the following for every $n, m \in \mathbb{N}$:

- (1) if $f(n) = m$, then $T \vdash \psi(\bar{n}, \bar{m})$; and
- (2) if $f(n) \neq m$, then $T \vdash \neg\psi(\bar{n}, \bar{m})$.

Definition 3.3 (Strongly Representable Functions). A function $f: \mathbb{N} \rightarrow \mathbb{N}$ is *strongly representable* in a theory T , if for some formula $\theta(x, y)$ we have the following for every $n, m \in \mathbb{N}$:

- (1) if $f(n) = m$, then $T \vdash \theta(\bar{n}, \bar{m})$; and
- (2) $T \vdash \forall y, z (\theta(\bar{n}, y) \wedge \theta(\bar{n}, z) \rightarrow y = z)$.

Definition 3.4 (Provably Total Functions). A function $f: \mathbb{N} \rightarrow \mathbb{N}$ is *provably total* in a theory T , if for some formula $\eta(x, y)$ we have the following for every $n, m \in \mathbb{N}$:

- (1) if $f(n) = m$, then $T \vdash \eta(\bar{n}, \bar{m})$; and
- (2) $T \vdash \forall x \exists y (\eta(x, y) \wedge \forall z [\eta(x, z) \rightarrow y = z])$.

Indeed, these definitions get stronger from top to bottom: If T is consistent and can prove $\bar{i} \neq \bar{j}$ for every distinct $i, j \in \mathbb{N}$, then every provably total function is strongly representable, and every strongly representable function is representable, and every representable function is weakly representable in T with the same formula. It is a folklore result that representability implies strong representability (cf. [15, Proposition I.3.3]):

Lemma 3.5 (Representability \implies Strong Representability). *If T proves $\forall y (y \neq 0), \forall y (y < \bar{n} \vee y = \bar{n} \vee \bar{n} < y)$ and $\forall y (y < \bar{n} + 1 \leftrightarrow y = \bar{0} \vee \dots \vee y = \bar{n})$, for all $n \in \mathbb{N}$, then the representability of a function in T implies its strong representability in it.*

Proof. If f is representable by $\psi(x, y)$ in T , then let $\theta(x, y) = \psi(x, y) \wedge \forall z < y \neg \psi(x, z)$. We show that $T \vdash \theta(\bar{n}, \overline{f(n)})$ and $T \vdash \theta(\bar{n}, y) \rightarrow y = \overline{f(n)}$ hold for any $n \in \mathbb{N}$ as follows. Reason in T : If $z < \overline{f(n)}$, then if $f(n) = 0$ we have a contradiction, otherwise (if $f(n) \neq 0$) we have $z = \bar{i}$ for some $i < f(n)$. Of course for any such i we have $\neg \psi(\bar{n}, \bar{i})$; thus $\neg \psi(\bar{n}, z)$. If $\theta(\bar{n}, y)$ and $y \neq \overline{f(n)}$, then either $y < \overline{f(n)}$ or $\overline{f(n)} < y$. In the former case we have $y = \bar{i}$ for some $i < f(n)$ if $f(n) \neq 0$, otherwise $y < 0$ is a contradiction; and so by $\neg \psi(\bar{n}, \bar{i})$ we have $\neg \psi(\bar{n}, y)$, which is a contradiction with $\theta(\bar{n}, y)$. In the latter case, $\forall z < y \neg \psi(\bar{n}, z)$ implies $\neg \psi(\bar{n}, \overline{f(n)})$; a contradiction again. \square

The question if the strong representability implies the provable totality was mentioned open in the first edition (1964) of the classical book [13]. In 1965, Verena (Huber-)Dyson showed that the strong representability implies the provable totality [3], and as a result this was Exercise 3.35 in the second edition (1979) of that book, and Exercise 3.32 in the third edition (1987), attributed to V. H. Dyson. Then in the fourth (1997), the fifth (2009) and the sixth (2015) editions, this has been proved in Proposition 3.12, attributed to V. H. Dyson again.

Theorem 3.6 (Strong Representability \implies Provable Totality). *If a function is strongly representable in a theory, then it is provably total in that theory.*

Proof. Let us note that we do not put any condition on the theory T ; let f be strongly representable by θ in T . Let $\exists!u \mathcal{A}(u)$ be an abbreviation for the formula $\exists u(\mathcal{A}(u) \wedge \forall v[\mathcal{A}(v) \rightarrow v = u])$. Put

$$\eta(x, y) = [\exists!z \theta(x, z) \wedge \theta(x, y)] \vee [\neg \exists!z \theta(x, z) \wedge y = 0].$$

For any $n \in \mathbb{N}$ we have $T \vdash \exists!y \theta(\bar{n}, y)$; thus from $T \vdash \theta(\bar{n}, \overline{f(n)})$ we get $T \vdash \eta(\bar{n}, \overline{f(n)})$. Now, we show that $T \vdash \forall x \exists!y \eta(x, y)$. Reason inside T : If $\exists!z \theta(x, z)$, then that unique z which satisfies $\theta(x, z)$ also satisfies $\eta(x, z)$ and $\forall u[\eta(x, u) \rightarrow u = z]$, whence $\exists!y \eta(x, y)$. If $\neg \exists!z \theta(x, z)$, then $y = 0$ is the unique y that satisfies $\eta(x, y)$. \square

The above proof of (Huber-)Dyson appears also in [8, page 63], [11, Proposition 3.8] and [19, Proposition 9.4.2]. The following theorem is usually proved by showing that every weakly representable function is recursive and that every recursive function is (strongly) representable; see e.g. [15, Corollary I.7.8] or [17, Theorem 4.5]. Here we present a simpler proof.

Theorem 3.7 (Weak Representability \implies Representability). *For a theory T , suppose the formula $\text{Proof}_T(z, x)$ states that “ z is (the Gödel code of) the proof of a formula (with Gödel code) x in T ”, and suppose that T has the following properties:*

- (a) $T \vdash \bar{i} \neq \bar{j}$, $T \vdash \bar{n} \leq \bar{m}$ and $T \vdash \forall y(\bar{m} \leq y \rightarrow \bar{n} \leq y)$, for all natural numbers i, j, n, m with $i \neq j$ and $n \leq m$;
 - (b) $T \vdash \forall y(y \leq \bar{n} \vee \bar{n} \leq y)$, for all $n \in \mathbb{N}$;
 - (c) $T \vdash \forall y(y \leq \bar{n} \leftrightarrow \bigvee_{i=0}^n y = \bar{i})$, for all $n \in \mathbb{N}$;
 - (d) if k is the Gödel code of a proof of ϕ in T , then $T \vdash \text{Proof}_T(\bar{k}, \ulcorner \phi \urcorner)$;
 - (e) if k is not the Gödel code of a proof of ϕ in T , then $T \vdash \neg \text{Proof}_T(\bar{k}, \ulcorner \phi \urcorner)$.
- Then weak representability of a function implies its representability in T .

Proof. Suppose the function f is weakly representable by φ in T . For the (bounded provability) predicate $\varrho(z, x) = \exists u \leq z \text{Proof}_T(u, x)$, let

$$\psi(x, y) = \exists z [\varrho(z, \ulcorner \varphi(x, y) \urcorner) \wedge \forall y' \leq z [y' \neq y \rightarrow \neg \varrho(z, \ulcorner \varphi(x, y') \urcorner)]] .$$

For showing the representability of f by ψ in T we prove that:

- (1) $T \vdash \psi(\bar{n}, \overline{f(n)})$ for all $n \in \mathbb{N}$, and
 - (2) $T \vdash \neg \psi(\bar{n}, \bar{m})$ for all $n, m \in \mathbb{N}$ with $m \neq f(n)$.
- (1): Fix an $n \in \mathbb{N}$ and let $k \in \mathbb{N}$ be a Gödel code for the proof of $T \vdash \varphi(\bar{n}, \overline{f(n)})$; so, we have $f(n) \leq k$. By (d) above we have

$$T \vdash \text{Proof}_T(\bar{k}, \ulcorner \varphi(\bar{n}, \overline{f(n)}) \urcorner),$$

and so

$$T \vdash \varrho(\bar{k}, \ulcorner \varphi(\bar{n}, \overline{f(n)}) \urcorner)$$

by (a) above. Now, for any $i \in \mathbb{N}$ with $i \neq f(n)$ we have that $T \not\vdash \varphi(\bar{n}, \bar{i})$, and so by (e) above, for any $l \in \mathbb{N}$, we have $T \vdash \neg \text{Proof}_T(\bar{l}, \ulcorner \varphi(\bar{n}, \bar{i}) \urcorner)$. Thus, by (c) above, $T \vdash \neg \varrho(\bar{l}, \ulcorner \varphi(\bar{n}, \bar{i}) \urcorner)$. Reason in T : for any y' with $y' \leq \bar{k}$ and $y' \neq \overline{f(n)}$, by (c) above, we have $y' = \bar{j}$ for some $j \leq k$ with $j \neq f(n)$. For any such j we have $\neg \varrho(\bar{k}, \ulcorner \varphi(\bar{n}, \bar{j}) \urcorner)$; and so, by (c) above, the sentence $\forall y' \leq \bar{k} [y' \neq y \rightarrow \neg \varrho(\bar{k}, \ulcorner \varphi(\bar{n}, y') \urcorner)]$ holds. Thus, $\psi(\bar{n}, \overline{f(n)})$.

(2): Fix some $n, m \in \mathbb{N}$ with $m \neq f(n)$. Let us note that

$$\neg \psi(x, y) \equiv \forall z [\varrho(z, \ulcorner \varphi(x, y) \urcorner) \rightarrow \exists y' \leq z [y' \neq y \wedge \varrho(z, \ulcorner \varphi(x, y') \urcorner)]] .$$

For proving $T \vdash \neg \psi(\bar{n}, \bar{m})$ we show that

$$T \vdash \forall z [\varrho(z, \ulcorner \varphi(\bar{n}, \bar{m}) \urcorner) \rightarrow \overline{f(n)} \leq z \wedge \overline{f(n)} \neq \bar{m} \wedge \varrho(z, \ulcorner \varphi(\bar{n}, \overline{f(n)}) \urcorner)] .$$

Let $k \in \mathbb{N}$ be a Gödel code for the proof of $T \vdash \varphi(\bar{n}, \overline{f(n)})$; so, $f(n) \leq k$. Also, from $T \not\vdash \varphi(\bar{n}, \bar{m})$, by (e) above, we have $T \vdash \neg \varrho(\bar{l}, \ulcorner \varphi(\bar{n}, \bar{m}) \urcorner)$,

for any $l \in \mathbb{N}$. Reason in T : for any z , by (b) above, we have either **(2.i)** $z \leq \bar{k}$ or **(2.ii)** $\bar{k} \leq z$. **(2.i)**: If $z \leq \bar{k}$ then $z = \bar{i}$ for some $i \leq k$, by (c) above. Now, $\varrho(\bar{i}, \ulcorner \varphi(\bar{n}, \bar{m}) \urcorner) \rightarrow \overline{f(n)} \leq \bar{i} \wedge \overline{f(n)} \neq \bar{m} \wedge \varrho(\bar{i}, \ulcorner \varphi(\bar{n}, \overline{f(n)}) \urcorner)$ follows from $\neg \varrho(\bar{i}, \ulcorner \varphi(\bar{n}, \bar{m}) \urcorner)$; thus $\neg \psi(\bar{n}, \bar{m})$ holds. **(2.ii)**: If $\bar{k} \leq z$, then $\overline{f(n)} \leq z$, by (a) above, which also implies $\overline{f(n)} \neq \bar{m}$. On the other hand, $\text{Proof}_T(\bar{k}, \ulcorner \varphi(\bar{n}, \overline{f(n)}) \urcorner)$ holds and so $\exists u \leq z \text{Proof}_T(u, \ulcorner \varphi(\bar{n}, \overline{f(n)}) \urcorner)$, or equivalently $\varrho(z, \ulcorner \varphi(\bar{n}, \overline{f(n)}) \urcorner)$. Thus, $\neg \psi(\bar{n}, \bar{m})$ holds by the already proved statement $\overline{f(n)} \leq z \wedge \overline{f(n)} \neq \bar{m} \wedge \varrho(z, \ulcorner \varphi(\bar{n}, \overline{f(n)}) \urcorner)$. \square

Let us note that the (very weak) finitely axiomatizable Robinson's Arithmetic Q satisfies all the conditions (a,b,c,d,e) in Theorem 3.7.

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Research Institute for Fundamental Sciences
 University of Tabriz, 29 Bahman Boulevard
 P.O.Box 51666-16471, Tabriz, Iran
 salehipour@tabrizu.ac.ir