

Varieties of many-sorted recognizable sets

SAEED SALEHI

Department of Mathematical Sciences
University of Tabriz
51666 Tabriz, Iran
e-mail: saesal@gmail.com

and

MAGNUS STEINBY*

Department of Mathematics
University of Turku
20014 Turku, Finland
and
Turku Centre for Computer Science
e-mail: steinby@utu.fi

(Received: July 16, 2006 and in revised form: July 20, 2006)

Abstract. We consider varieties of recognizable subsets of many-sorted finitely generated free algebras over a given variety, varieties of congruences of such algebras, and varieties of finite many-sorted algebras. A variety theorem that establishes bijections between the classes of these three types of varieties is proved. For this, appropriate notions of many-sorted syntactic congruences and algebras are needed. Also an alternative type of varieties is considered where each subset consists of elements of just one sort.

Mathematics Subject Classifications (2000). 68Q70, 08A70, 08B15, 08A68

1 Introduction

S. Eilenberg's [8] famous Variety Theorem establishes a bijective correspondence between varieties of regular languages ($*$ -varieties) and varieties of finite monoids, or between varieties of regular languages without the empty word ($+$ -varieties) and varieties of finite semigroups. The theorem provides a general framework for the classification of regular languages and it describes the families of regular languages that can be characterized by syntactic monoids or by syntactic semigroups.

The Variety Theorem has been extended or adapted to other kinds of regular sets in several ways. A useful addition was the correspondence between varieties of regular languages and varieties of congruences of free monoids or free semigroups introduced by Thérien [25]. Another notable extension of the basic theory is Pin's [18] theory of positive varieties. In [21] Steinby proposes a theory of varieties of recognizable subsets of free algebras that encompasses both Eilenberg's theory and a theory of varieties of regular tree languages as special cases. The idea of recognizable subsets of arbitrary algebras goes back

*Corresponding author.

to Mezei and Wright [16]. The similar generalization developed in the more extensive study [1] by Almeida includes also varieties of congruences. Varieties of congruences appear also in the theory of varieties of tree languages presented in [22] and in the theory of generalized varieties of tree languages of [23]. The theories in [1, 21, 22, 23, 24] are all based on syntactic algebras. As one more extension along these lines, we should mention Ésik's theory [10] where the place of varieties of finite algebras is taken by varieties of finitary theories.

It appears that Maibaum [14] was the first one to consider many-sorted tree languages. Many-sorted trees are used also by Engelfriet and Schmidt [9] in their study of the equational semantics of context-free tree languages. Recognizable subsets of general many-sorted algebras were studied by Courcelle [5, 6].

In this paper we join two of the above lines of research by developing a theory of varieties of recognizable subsets of free many-sorted algebras. Thus we actually generalize the theories of [21, 22, 24] and [1] to the many-sorted case. It should be mentioned that, although not considered here, Wilke's [26] tree algebras gave an important impetus to this work; they are 3-sorted algebras used for characterizing families of (binary) tree languages. These algebras we studied in [20] (see also [19]). The syntactic preclones recently introduced by Ésik and Weil [11] also lead to a many-sorted formalism, but it remains to be studied how our results apply to their theory.

In Section 2 we present some basic definitions and our notation for many-sorted algebras. Also some more specialized notions relevant to our work are introduced. The references [13] and [15] may be consulted for general treatments of the theory of many-sorted algebras. In Section 3 recognizable subsets of many-sorted algebras are considered. There are actually two types of these subsets: recognizable sorted subsets, and the "pure" recognizable subsets considered in [6, 9, 14] in which all elements are of some given sort. We mainly consider the former type but we will show how the theory applies also to the other kind of sets.

Syntactic congruences and syntactic algebras of subsets of many-sorted algebras are introduced in Section 4, and it is shown that they enjoy all the same general properties as their counterparts for monoids [8, 17] or term algebras, or one-sorted algebras in general [1, 21, 22].

In Section 5 we define our varieties of recognizable sets and varieties of congruences. For this a finite set of sorts S and variety \mathbf{V} of some finite S -sorted type Ω are fixed. A variety of recognizable \mathbf{V} -sets consists then of recognizable subsets of the finitely generated free algebras over \mathbf{V} . Similarly, a variety of \mathbf{V} -congruences consists of congruences of finite index on these algebras. Finally, a \mathbf{V} -variety of finite algebras is defined as a variety of finite algebras contained in \mathbf{V} . In Section 6 we define six mappings that transform varieties of recognizable \mathbf{V} -sets, varieties of \mathbf{V} -congruences and \mathbf{V} -varieties of finite algebras to each other. Then we prove our Variety Theorem that essentially says that these six mappings form three pairs of mutually inverse isomorphisms between the complete lattices of the three kinds of varieties considered. The proof is very similar to the corresponding proof presented in [22], but there are naturally some technical differences and for the reader's convenience a rather detailed

proof is presented.

In Section 7 we define varieties of pure recognizable \mathbf{V} -sets in which each recognizable set is a subset of the set of elements of some given sort of a finitely generated free algebra over \mathbf{V} . By establishing a natural correspondence between the two types of varieties of recognizable \mathbf{V} -sets, a Variety Theorem is derived also for varieties of pure recognizable \mathbf{V} -sets.

2 Many-sorted algebras

In this section we review some basic concepts related to many-sorted (or heterogeneous) algebras, and at the same time we fix our notation and introduce a few more special notions. For more about many-sorted algebras and their applications to automata and formal languages, the reader may consult [12, 2, 14, 13, 5, 6], for example, or the survey [15] that also contains many further references.

In what follows, S is always a non-empty set of *sorts*. Families of objects indexed by S are said to be S -sorted, or just *sorted*. The sort of an object is usually shown as a subscript or in parentheses (to avoid multiple subscripts). For example, an S -sorted set $A = \langle A_s \rangle_{s \in S}$ is an S -indexed family of sets, where for each $s \in S$, A_s is the set of *elements of sort s* in A , and we may write A_s also as $A(s)$. The basic set-theoretic notions are defined for S -sorted sets in the natural sortwise manner: for any S -sorted sets $A = \langle A_s \rangle_{s \in S}$ and $B = \langle B_s \rangle_{s \in S}$, $A \subseteq B$ means that $A_s \subseteq B_s$ for every $s \in S$, $A \cup B = \langle A_s \cup B_s \rangle_{s \in S}$ etc., and general sorted unions and intersections are defined similarly. The notation \emptyset is used also for the S -sorted empty set $\langle \emptyset \rangle_{s \in S}$.

Sometimes we form a sorted set from a set of a given sort: with any set T of some sort $u \in S$ we associate the S -sorted set $\langle T \rangle$ such that $\langle T \rangle_u = T$ and $\langle T \rangle_s = \emptyset$ for every $s \in S - \{u\}$.

A *sorted relation* $\theta = \langle \theta_s \rangle_{s \in S}$ on $A = \langle A_s \rangle_{s \in S}$ is a family of relations such that for each $s \in S$, θ_s is a relation on A_s , and it is a *sorted equivalence* on A if θ_s is an equivalence on A_s for every $s \in S$. Let $\text{Eq}_S(A)$ denote the set of all sorted equivalences on A . For any $\theta = \langle \theta_s \rangle_{s \in S} \in \text{Eq}_S(A)$, the *quotient set* is the S -sorted set $A/\theta = \langle A_s/\theta_s \rangle_{s \in S}$, where $A_s/\theta_s = \{a/\theta_s \mid a \in A_s\}$ ($s \in S$). Of course, $\text{Eq}_S(A)$ forms with respect to the sorted inclusion relation a complete lattice in which least upper bounds and greatest lower bounds are formed sortwise. The least element is the *sorted diagonal relation* $\Delta_A = \langle \Delta_{A(s)} \rangle_{s \in S}$ and the greatest element is the *sorted universal relation* $\nabla_A = \langle \nabla_{A(s)} \rangle_{s \in S}$, where $\Delta_{A(s)} = \{(a, a) \mid a \in A(s)\}$ and $\nabla_{A(s)} = A(s) \times A(s)$ for each $s \in S$.

A *sorted mapping* $\varphi: A \rightarrow B$ from $A = \langle A_s \rangle_{s \in S}$ to $B = \langle B_s \rangle_{s \in S}$ is an S -sorted family $\varphi = \langle \varphi_s \rangle_{s \in S}$ of mappings $\varphi_s: A_s \rightarrow B_s$ ($s \in S$). The *kernel* of φ is the sorted equivalence $\ker \varphi = \langle \ker \varphi_s \rangle_{s \in S}$ on A . For any sorted subset $H \subseteq A$, $H\varphi$ denotes the sorted subset $\langle H_s\varphi_s \rangle_{s \in S}$ of B . Similarly, if H is a sorted subset of B , then $H\varphi^{-1}$ denotes the sorted subset $\langle H_s\varphi_s^{-1} \rangle_{s \in S}$ of A . The *composition* of two S -sorted mappings $\varphi: A \rightarrow B$ and $\psi: B \rightarrow C$ is the sorted mapping $\varphi\psi: A \rightarrow C$ such that $(\varphi\psi)_s = \varphi_s\psi_s$ for each $s \in S$. Here the mappings were

composed from left to right, as we shall do especially with homomorphisms. Hence, $\varphi_s \psi_s: a \mapsto (a\varphi_s)\psi_s$ for all $s \in S$ and $a \in A_s$.

Treating S as an alphabet, S^* denotes the set of finite strings over S , including the empty string e , and S^+ is the set of non-empty strings over S . An S -sorted signature Ω is a set of operation symbols each of which has a *type* that is an element of $S^* \times S$. For any $(w, s) \in S^* \times S$, let $\Omega_{w,s}$ be the set of symbols of type (w, s) in Ω . If $f \in \Omega_{w,s}$, then w is the *domain type* of f , and s is its *sort*. Elements of $\Omega_{e,s}$ are *constant symbols* of sort s . The fact that $f \in \Omega_{w,s}$ is expressed also by writing $f: w \rightarrow s$. For S finite, a finite S -sorted signature is called an S -sorted ranked alphabet. Later S will always be finite and Ω will be an S -sorted ranked alphabet.

An Ω -algebra $\mathcal{A} = (A, \Omega)$ consists of an S -sorted set $A = \langle A_s \rangle_{s \in S}$, where $A_s \neq \emptyset$ for every $s \in S$, and

- (1) for any $c \in \Omega_{e,s}$ with $s \in S$, a constant $c^{\mathcal{A}} \in A_s$ of sort s is specified;
- (2) for any $f \in \Omega_{w,s}$ with $(w, s) \in (S^+, S)$, an operation $f^{\mathcal{A}}: A^w \rightarrow A_s$ of type (w, s) , domain type w and sort s is defined. Here $A^w = A_{s(1)} \times \cdots \times A_{s(m)}$ assuming that $w = s(1) \dots s(m)$.

Such an algebra \mathcal{A} is said to be S -sorted. When we speak about the Ω -algebras $\mathcal{A} = (A, \Omega)$, $\mathcal{B} = (B, \Omega)$ and $\mathcal{C} = (C, \Omega)$, we usually assume without mentioning it that $A = \langle A_s \rangle_{s \in S}$, $B = \langle B_s \rangle_{s \in S}$ and $C = \langle C_s \rangle_{s \in S}$.

An Ω -algebra $\mathcal{B} = (B, \Omega)$ such that $B \subseteq A$ is a *subalgebra* of $\mathcal{A} = (A, \Omega)$, if $c^{\mathcal{B}} = c^{\mathcal{A}}$ for every constant symbol $c \in \Omega_{e,s}$ ($s \in S$), and $f^{\mathcal{B}} = f^{\mathcal{A}}|_{B^w}$ for any $f \in \Omega_{w,s}$ with $w \in S^+$ and $s \in S$. If \mathcal{B} is a subalgebra of \mathcal{A} , then $B = \langle B_s \rangle_{s \in S}$ is a *closed subset* of \mathcal{A} , that is, $c^{\mathcal{A}} \in B_s$ whenever $c \in \Omega_{e,s}$ and $s \in S$, and $f^{\mathcal{A}}(b_1, \dots, b_m) \in B_s$ for $f: s(1) \dots s(m) \rightarrow s$ and $b_1 \in B_{s(1)}, \dots, b_m \in B_{s(m)}$.

On the other hand, any closed subset B with $B_s \neq \emptyset$ for every $s \in S$, is the carrier set of a unique subalgebra of \mathcal{A} . Hence, subalgebras coincide with the closed subsets whose all components are non-empty. The set of all closed subsets of \mathcal{A} is denoted by $\text{Sub}(\mathcal{A})$, and let $\text{Sub}^+(\mathcal{A})$ denote the set of closed subsets with non-empty components. Any subset $H = \langle H_s \rangle_{s \in S}$ of A is contained in a unique minimal closed subset $[H] = \bigcap \{B \mid H \subseteq B, B \in \text{Sub}(\mathcal{A})\}$, the *closed subset generated* by H . If $H_s \cup \Omega_{e,s} \neq \emptyset$ for every $s \in S$, then $[H]$ is a subalgebra, but this may be the case even otherwise. If $[H] \in \text{Sub}^+(\mathcal{A})$, then $[H]$ is called the *subalgebra generated* by H .

A sorted equivalence $\theta = \langle \theta_s \rangle_{s \in S}$ on A is a *congruence* on $\mathcal{A} = (A, \Omega)$ if

$$a_1 \theta_{s(1)} b_1, \dots, a_m \theta_{s(m)} b_m \Rightarrow f^{\mathcal{A}}(a_1, \dots, a_m) \theta_s f^{\mathcal{A}}(b_1, \dots, b_m),$$

whenever $f: s(1) \dots s(m) \rightarrow s$ and $a_1, b_1 \in A_{s(1)}, \dots, a_m, b_m \in A_{s(m)}$. Then the operations of the *quotient algebra* $\mathcal{A}/\theta = (A/\theta, \Omega)$ are well-defined by setting

- (1) $c^{\mathcal{A}/\theta} = c^{\mathcal{A}}/\theta_s$ for every $c \in \Omega_{e,s}$, and
- (2) $f^{\mathcal{A}/\theta}(a_1/\theta_{s(1)}, \dots, a_m/\theta_{s(m)}) = f^{\mathcal{A}}(a_1, \dots, a_m)/\theta_s$ for any function symbol $f: s(1) \dots s(m) \rightarrow s$ and any $a_1 \in A_{s(1)}, \dots, a_m \in A_{s(m)}$.

A sorted mapping $\varphi: A \rightarrow B$ is a *homomorphism* from $\mathcal{A} = (A, \Omega)$ to $\mathcal{B} = (B, \Omega)$, and we express this by writing $\varphi: \mathcal{A} \rightarrow \mathcal{B}$, if

- (1) $c^{\mathcal{A}}\varphi_s = c^{\mathcal{B}}$ whenever $c \in \Omega_{e,s}$ for some $s \in S$, and
- (2) $f^{\mathcal{A}}(a_1, \dots, a_m)\varphi_s = f^{\mathcal{B}}(a_1\varphi_{s(1)}, \dots, a_m\varphi_{s(m)})$ for any function symbol $f: s(1) \dots s(m) \rightarrow s$ and any $a_1 \in A_{s(1)}, \dots, a_m \in A_{s(m)}$.

A homomorphism φ is a *monomorphism*, an *epimorphism* or an *isomorphism*, if every φ_s ($s \in S$) is injective, surjective or bijective, respectively. If there is an isomorphism $\varphi: \mathcal{A} \rightarrow \mathcal{B}$, the algebras are *isomorphic*, $\mathcal{A} \cong \mathcal{B}$ in symbols. If there exists an epimorphism $\varphi: \mathcal{A} \rightarrow \mathcal{B}$, then \mathcal{B} is an *image* of \mathcal{A} , and we write $\mathcal{B} \leftarrow \mathcal{A}$. The fact that \mathcal{B} is isomorphic to a subalgebra of \mathcal{A} is expressed writing $\mathcal{B} \subseteq \mathcal{A}$. Furthermore, \mathcal{B} is said to *divide* \mathcal{A} , and we write $\mathcal{B} \preceq \mathcal{A}$, if \mathcal{B} is an image of a subalgebra of \mathcal{A} . Clearly, $\mathcal{B} \preceq \mathcal{A}$ iff there is an Ω -algebra \mathcal{C} for which there exist a monomorphism $\varphi: \mathcal{C} \rightarrow \mathcal{A}$ and an epimorphism $\psi: \mathcal{C} \rightarrow \mathcal{B}$, and $\mathcal{B} \preceq \mathcal{A}$ follows both from $\mathcal{B} \subseteq \mathcal{A}$ and from $\mathcal{B} \leftarrow \mathcal{A}$.

The *natural map* corresponding to a sorted equivalence $\theta = \langle \theta_s \rangle_{s \in S}$ on a sorted set A , is the sorted map $\theta^\sharp: A \rightarrow A/\theta$, where $\theta_s^\sharp: A_s \rightarrow A_s/\theta_s, a \mapsto a/\theta_s$, for each $s \in S$. If θ is a congruence on an Ω -algebra \mathcal{A} , then θ^\sharp is an epimorphism from \mathcal{A} onto \mathcal{A}/θ , and the Homomorphism Theorem (cf. [3], for example) can be generalized to many-sorted algebras as follows (cf. [15], for example).

PROPOSITION 2.1 *If $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ is a homomorphism of Ω -algebras, then $\ker \varphi$ is a congruence on \mathcal{A} and $\psi: \mathcal{A}/\ker \varphi \rightarrow \mathcal{B}, a/\ker \varphi_s \mapsto a\varphi_s$, is a monomorphism such that $(\ker \varphi)^\sharp \psi = \varphi$. If φ is an epimorphism, then ψ is an isomorphism. \square*

Next we introduce the many-sorted version of a notion that has proved very useful for dealing with congruences.

Let $\mathcal{A} = (A, \Omega)$ be an Ω -algebra. For any pair $s, s' \in S$ of sorts, an *elementary s, s' -translation* is any mapping $A_s \rightarrow A_{s'}$ of the form

$$\alpha(\xi_s) = f^{\mathcal{A}}(a_1, \dots, a_{j-1}, \xi_s, a_{j+1}, \dots, a_m),$$

where $m \geq 1, f: s(1) \dots s(m) \rightarrow s', 1 \leq j \leq m, s(j) = s$, and $a_i \in A_{s(i)}$ for every $i \neq j$. Here and later, ξ_s is a variable of sort s that does not appear in the other alphabets considered. Let the set of all elementary s, s' -translations of \mathcal{A} be denoted by $\text{ETr}(\mathcal{A}, s, s')$. The $S \times S$ -sorted set $\text{Tr}(\mathcal{A}) = \langle \text{Tr}(\mathcal{A}, s, s') \rangle_{s, s' \in S}$ of all *translations* of \mathcal{A} is now defined inductively by the following clauses:

- (1) $\text{ETr}(\mathcal{A}, s, s') \subseteq \text{Tr}(\mathcal{A}, s, s')$ for all $s, s' \in S$,
- (2) for each $s \in S$, the identity map $1_{A(s)}: A(s) \rightarrow A(s)$ is in $\text{Tr}(\mathcal{A}, s, s)$, and
- (3) if $\alpha(\xi_s) \in \text{Tr}(\mathcal{A}, s, s')$ and $\beta(\xi_{s'}) \in \text{Tr}(\mathcal{A}, s', s'')$, for some $s, s', s'' \in S$, then $\beta(\alpha(\xi_s)) \in \text{Tr}(\mathcal{A}, s, s'')$.

For any $s, s' \in S$, the elements of $\text{Tr}(\mathcal{A}, s, s')$ are the *s, s' -translations* of \mathcal{A} .

The following lemma is an immediate generalization of the corresponding fact about one-sorted algebras (see e.g. [3, 4, 7]).

LEMMA 2.2 *Let $\mathcal{A} = (A, \Omega)$ be an Ω -algebra. Every congruence $\theta = \langle \theta_s \rangle_{s \in S}$ on \mathcal{A} is invariant with respect to all translations of \mathcal{A} , that is to say, $a \theta_s b$ implies $\alpha(a) \theta_{s'} \alpha(b)$ for all $s, s' \in S$, $a, b \in A_s$ and $\alpha(\xi_s) \in \text{Tr}(\mathcal{A}, s, s')$. On the other hand, a sorted equivalence θ on A is a congruence if it is invariant with respect to every elementary translation of \mathcal{A} . \square*

The following lemma is often used (cf. [21, 22] for the one-sorted version).

LEMMA 2.3 *Let $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ be a homomorphism of Ω -algebras from $\mathcal{A} = (A, \Omega)$ to $\mathcal{B} = (B, \Omega)$. For any $s, s' \in S$ and every $\alpha(\xi_s)$ in $\text{Tr}(\mathcal{A}, s, s')$, there exists a translation $\alpha_\varphi(\xi_s) \in \text{Tr}(\mathcal{B}, s, s')$ such that*

$$\alpha(a)\varphi_{s'} = \alpha_\varphi(a\varphi_s)$$

for every $a \in A_s$. If φ is an epimorphism, then for all $s, s' \in S$ and every $\beta(\xi_s)$ in $\text{Tr}(\mathcal{B}, s, s')$ there exists an $\alpha(\xi_s) \in \text{Tr}(\mathcal{A}, s, s')$ such that $\beta = \alpha_\varphi$. \square

Translations of an Ω -algebra $\mathcal{A} = (A, \Omega)$ and their inverses are applied to subsets of a given sort and to sorted subsets as follows. Let $\alpha(\xi_s) \in \text{Tr}(\mathcal{A}, s, s')$ for some $s, s' \in S$. For any $u \in S$ and $T \subseteq A_u$, let

- $\alpha(T) = \{\alpha(a) \mid a \in T\} (\subseteq A_{s'})$ if $u = s$, and $\alpha(T) = \emptyset$ if $u \neq s$;
- $\alpha^{-1}(T) = \{a \in A_s \mid \alpha(a) \in T\}$ if $u = s'$, and $\alpha^{-1}(T) = \emptyset$ if $u \neq s'$.

Furthermore, for any sorted subset $L = \langle L_s \rangle_{s \in S}$ of A , we set

- $\alpha(L) = \langle K_u \rangle_{u \in S}$, where $K_{s'} = \alpha(L_s)$, and $K_u = \emptyset$ for each $u \neq s'$, and
- $\alpha^{-1}(L) = \langle K_u \rangle_{u \in S}$, where $K_s = \alpha^{-1}(L_{s'})$, and $K_u = \emptyset$ for each $u \neq s$.

The direct product $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ of any finite family $\mathcal{A}_1, \dots, \mathcal{A}_n$ Ω -algebras, or the direct product $\prod_{i \in I} \mathcal{A}_i$ of a general family \mathcal{A}_i ($i \in I$) of Ω -algebras, are defined in the natural way.

If $\varphi: A \rightarrow B$ is a sorted mapping from an S -sorted set $A = \langle A_s \rangle_{s \in S}$ to an S -sorted set $B = \langle B_s \rangle_{s \in S}$ and $\theta = \langle \theta_s \rangle_{s \in S}$ is a sorted equivalence on B , then $\varphi \circ \theta \circ \varphi^{-1}$ is the sorted equivalence on A defined by the condition

$$a_1 (\varphi \circ \theta \circ \varphi^{-1})_s a_2 \Leftrightarrow a_1 \varphi_s \theta_s a_2 \varphi_s \quad (s \in S, a_1, a_2 \in A_s).$$

The following facts are again simple generalizations from the one-sorted case.

LEMMA 2.4 *Let $\mathcal{A} = (A, \Omega)$ and $\mathcal{B} = (B, \Omega)$ be Ω -algebras, θ and θ' be congruences on \mathcal{A} , ρ be a congruence on \mathcal{B} , and let $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ be a homomorphism.*

- (1) *If $\theta \subseteq \theta'$, then $\mathcal{A}/\theta' \leftarrow \mathcal{A}/\theta$.*
- (2) *$\mathcal{A}/\theta \cap \theta' \subseteq \mathcal{A}/\theta \times \mathcal{A}/\theta'$.*
- (3) *The relation $\varphi \circ \rho \circ \varphi^{-1}$ is a congruence on \mathcal{A} , and $\mathcal{A}/\varphi \circ \rho \circ \varphi^{-1} \preceq \mathcal{B}/\rho$. Moreover, if φ is an epimorphism, then $\mathcal{A}/\varphi \circ \rho \circ \varphi^{-1} \cong \mathcal{B}/\rho$. \square*

The class operators S , H , P and P_f are defined exactly as in the one-sorted case: for any class \mathbf{K} of Ω -algebras, $S(\mathbf{K})$ is the class of algebras isomorphic to a subalgebra of a member of \mathbf{K} , $H(\mathbf{K})$ is the class of all the images of members of \mathbf{K} , $P(\mathbf{K})$ consists of all the algebras isomorphic to the direct product of a family of algebras in \mathbf{K} , and $P_f(\mathbf{K})$ is the class of algebras isomorphic to the direct product of a finite family of algebras in \mathbf{K} .

A class \mathbf{K} of Ω -algebras is a *variety* if $S(\mathbf{K}), H(\mathbf{K}), P(\mathbf{K}) \subseteq \mathbf{K}$. Birkhoff's well-known theorem by which a class of algebras is definable by equations iff it is a variety, holds also for many-sorted algebras (cf. Section 5 of [15]).

A class \mathbf{K} of finite Ω -algebras is called a *variety of finite Ω -algebras*, an Ω -VFA for short, if $S(\mathbf{K}), H(\mathbf{K}), P_f(\mathbf{K}) \subseteq \mathbf{K}$. It is easy to show that a class \mathbf{K} of finite Ω -algebras is an Ω -VFA iff $\mathcal{A} \in \mathbf{K}$ whenever $\mathcal{A} \preceq \mathcal{A}_1 \times \dots \times \mathcal{A}_n$ for some $n \geq 0$ and $\mathcal{A}_1, \dots, \mathcal{A}_n \in \mathbf{K}$. When we deal with varieties of finite Ω -algebras, both S and Ω are assumed to be finite.

Let $X = \langle X_s \rangle_{s \in S}$ be an S -sorted alphabet disjoint from Ω . The S -sorted set $T_\Omega(X) = \langle T_\Omega(X, s) \rangle_{s \in S}$ of Ω -terms with variables in X is defined inductively:

- (1) $\Omega_{e,s} \cup X_s \subseteq T_\Omega(X, s)$ for every $s \in S$, and
- (2) $f(t_1, \dots, t_m) \in T_\Omega(X, s)$ for any function symbol $f: s_1 \dots s_m \rightarrow s$ ($m > 0$) and any terms $t_1 \in T_\Omega(X, s_1), \dots, t_m \in T_\Omega(X, s_m)$.

The alphabet X is said to be *full* for Ω if $T_\Omega(X, s) \neq \emptyset$ for every $s \in S$. Note that a given $T_\Omega(X, s)$ may be non-empty even if $X_s = \Omega_{e,s} = \emptyset$. If X is full for Ω , then the ΩX -term algebra $\mathcal{T}_\Omega(X) = (T_\Omega(X), \Omega)$ can be defined thus:

- (1) $c^{\mathcal{T}_\Omega(X)} = c$ for any $s \in S$ and $c \in \Omega_{e,s}$, and
- (2) $f^{\mathcal{T}_\Omega(X)}(t_1, \dots, t_m) = f(t_1, \dots, t_m)$ for any $f: s_1 \dots s_m \rightarrow s$ ($m > 0$) and any $t_1 \in T_\Omega(X, s_1), \dots, t_m \in T_\Omega(X, s_m)$.

Of course, $\mathcal{T}_\Omega(X)$ is *generated freely* by X over the class of all Ω -algebras, that is to say, for any Ω -algebra $\mathcal{A} = (A, \Omega)$, any sorted mapping $\alpha: X \rightarrow A$ has a unique extension to a homomorphism $\alpha^{\mathcal{A}}: \mathcal{T}_\Omega(X) \rightarrow \mathcal{A}$.

More generally, if \mathbf{V} is a class of Ω -algebras, an Ω -algebra \mathcal{F} is *generated freely* over \mathbf{V} by a sorted subset G , if $\mathcal{F} \in \mathbf{V}$, \mathcal{F} is generated by G , and for any $\mathcal{A} = (A, \Omega)$ in \mathbf{V} , any sorted mapping $\varphi_0: G \rightarrow A$ can be extended to a homomorphism $\varphi: \mathcal{F} \rightarrow \mathcal{A}$. If such an \mathcal{F} exists, it is unique up to isomorphism, and we denote it by $\mathcal{F}_{\mathbf{V}}(G) = (F_{\mathbf{V}}(G), \Omega)$ with $F_{\mathbf{V}}(G) = \langle F_{\mathbf{V}}(G, s) \rangle_{s \in S}$.

Let Ω be an S -sorted ranked alphabet and let X be an S -sorted alphabet disjoint from Ω . For each $s \in S$, let ξ_s be again a special symbol of sort s . The $S \times S$ -sorted set $C_\Omega(X) = \langle C_\Omega(X, s, s') \rangle_{s, s' \in S}$ of ΩX -contexts is defined inductively by the conditions

- (1) $\xi_s \in C_\Omega(X, s, s)$ for each $s \in S$, and
- (2) $f(t_1, \dots, t_{j-1}, p, t_{j+1}, \dots, t_m) \in C_\Omega(X, s, s')$ whenever $s, s', s_1, \dots, s_m \in S$, $f: s_1 \dots s_m \rightarrow s'$, $1 \leq j \leq m$, $p \in C_\Omega(X, s, s_j)$, and $t_i \in T_\Omega(X, s_i)$, $i \neq j$.

The *product* $p \cdot q = q(p)$ of $p \in C_\Omega(X, s, s')$ and $q \in C_\Omega(X, s', s'')$ ($s, s', s'' \in S$) is the ΩX -context in $C_\Omega(X, s, s'')$ obtained from q when $\xi_{s'}$ is replaced with p .

Let $\mathcal{A} = (A, \Omega)$ be any Ω -algebra. Every translation of \mathcal{A} is *represented* in a natural way by an ΩA -context of a matching type:

- (1) an elementary translation $\alpha(\xi_s) = f^A(a_1, \dots, a_{j-1}, \xi_s, a_{j+1}, \dots, a_m)$ is represented by the ΩA -context $f(a_1, \dots, a_{j-1}, \xi_s, a_{j+1}, \dots, a_m)$,
- (2) the identity map $1_{A(s)}: A(s) \rightarrow A(s)$ is represented by ξ_s , and
- (3) if $\alpha(\xi_s) \in \text{Tr}(\mathcal{A}, s, s')$ and $\beta(\xi_{s'}) \in \text{Tr}(\mathcal{A}, s', s'')$ are represented by the contexts $p(\xi_s) \in C_\Omega(A, s, s')$ and $q(\xi_{s'}) \in C_\Omega(A, s', s'')$, respectively, then $\beta(\alpha(\xi_s))$ is represented by $q(p(\xi_s)) \in C_\Omega(A, s, s'')$.

3 Recognizable subsets

An equivalence θ on a set A *saturates* a subset $L \subseteq A$ when L is the union of some θ -classes, and that θ is said to be of *finite index* if it has a finite number of equivalence classes. Mezei and Wright [16] call a subset L of an algebra \mathcal{A} *recognizable* if it is saturated by a congruence of finite index on \mathcal{A} . Clearly, L is recognizable iff there exist a finite algebra \mathcal{B} , a homomorphism $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ and a subset H of \mathcal{B} such that $L = H\varphi^{-1}$. We use this condition, where \mathcal{B} may be viewed as a ‘recognizer’ of L , for defining recognizability in many-sorted algebras. There are two natural types of recognizable subsets of a sorted algebra: the *recognizable sorted subsets* and the *recognizable subsets of a given sort*.

In what follows, S is always a finite set of sorts and Ω is an S -sorted ranked alphabet. An S -sorted set $A = \langle A_s \rangle_{s \in S}$ is said to be *finite* if every A_s ($s \in S$) is finite, and an Ω -algebra $\mathcal{A} = (A, \Omega)$ is *finite* if $A = \langle A_s \rangle_{s \in S}$ is finite.

DEFINITION 3.1 A sorted subset $L \subseteq A$ of an Ω -algebra $\mathcal{A} = (A, \Omega)$ is *recognizable* if there exist a finite Ω -algebra $\mathcal{B} = (B, \Omega)$, a homomorphism $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ and a sorted subset H of B such that $L = H\varphi^{-1}$. Then we say also that \mathcal{B} *recognizes* L . Let $\text{Rec}(\mathcal{A})$ denote the set of all recognizable subsets of \mathcal{A} .

For any $s \in S$, a subset T of A_s is said to be *recognizable* in \mathcal{A} if there exist a finite Ω -algebra $\mathcal{B} = (B, \Omega)$, a homomorphism $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ and a subset H of B_s such that $T = H\varphi_s^{-1}$. Let $\text{Rec}(\mathcal{A}, s)$ denote the set of all such subsets of A_s . We call such sets also *pure recognizable sets*.

The recognizable tree languages of sort $s \in S$ considered by Maibaum [14] are the pure recognizable subsets of the term algebra $\mathcal{T}_\Omega(\emptyset)$ of sort s , i.e., the elements of $\text{Rec}(\mathcal{T}_\Omega(\emptyset), s)$. Courcelle [5, 6] extends this notion to any S -sorted algebra $\mathcal{A} = (\langle A_s \rangle_{s \in S}, \Omega)$, without assuming the finiteness of S or Ω , by calling a subset $T \subseteq A_s$ recognizable if there exist a ‘locally finite’ Ω -algebra $\mathcal{B} = (B, \Omega)$, a homomorphism $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ and a subset H of B_s such that $T = H\varphi_s^{-1}$; in [5, 6] an algebra $\mathcal{B} = (\langle B_s \rangle_{s \in S}, \Omega)$ is called *locally finite* if every B_s is finite ($s \in S$). Since we assume that S is finite, this ‘locally finite’ means here just ‘finite’, and hence our pure recognizable subsets are exactly Courcelle’s recognizable subsets.

Although we are primarily concerned with sorted recognizable sets, we will also note how the theory can be adapted to pure recognizable sets.

A sorted equivalence $\theta = \langle \theta_s \rangle_{s \in S}$ on an S -sorted set $A = \langle A_s \rangle_{s \in S}$ is said to *saturate* a sorted subset $L = \langle L_s \rangle_{s \in S}$ of A if every L_s is the union of some θ_s -classes ($s \in S$), and θ is of *finite index* if every θ_s ($s \in S$) is of finite index. The following lemma is an obvious generalization of the fact noted above.

LEMMA 3.2 *A sorted subset of an Ω -algebra \mathcal{A} is recognizable iff it is saturated by a congruence on \mathcal{A} of finite index. Similarly, a subset $T \subseteq A_u$ of some sort $u \in S$ is recognizable iff it is saturated by θ_u for some congruence $\theta = \langle \theta_s \rangle_{s \in S}$ on \mathcal{A} of finite index. \square*

Next we present a few closure properties that are well-known for recognizable subsets of one-sorted algebras.

PROPOSITION 3.3 *Let $\mathcal{A} = (A, \Omega)$ and $\mathcal{B} = (B, \Omega)$ be any Ω -algebras.*

- (1) $\emptyset, A \in \text{Rec}(\mathcal{A})$.
- (2) *If $K, L \in \text{Rec}(\mathcal{A})$, then $K \cup L, K \cap L, K - L \in \text{Rec}(\mathcal{A})$.*
- (3) *If $L \in \text{Rec}(\mathcal{A})$ and $\alpha \in \text{Tr}(\mathcal{A}, s, s')$ for some $s, s' \in S$, then $\alpha^{-1}(L) \in \text{Rec}(\mathcal{A})$.*
- (4) *If $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ is a homomorphism and $L \in \text{Rec}(\mathcal{B})$, then $L\varphi^{-1} \in \text{Rec}(\mathcal{A})$.*

Proof. Assertion (1) is trivial, and (2) can be proved as usual by defining the direct product of any two finite algebras recognizing K and L , respectively.

For (3), we recall first that $\alpha^{-1}(L)_s = \alpha^{-1}(L_{s'})$ and $\alpha^{-1}(L)_{s''} = \emptyset$ for every $s'' \neq s$. Assume now that $L = H\varphi^{-1}$, where $\varphi: \mathcal{A} \rightarrow \mathcal{C}$ is a homomorphism to a finite Ω -algebra $\mathcal{C} = (C, \Omega)$, and $H \subseteq C$. By Lemma 2.3 there is a translation $\alpha_\varphi \in \text{Tr}(\mathcal{C}, s, s')$ such that $\alpha(a)\varphi_{s'} = \alpha_\varphi(a\varphi_s)$ for every $a \in L_s$. Now it is easy to see that $\alpha^{-1}(L) = G\varphi^{-1}$ for the sorted subset G of C defined in such a way that $G_s = \alpha_\varphi^{-1}(H_{s'})$ and $G_{s''} = \emptyset$ for every $s'' \neq s$.

To prove (4), assume that $L = H\psi^{-1}$, where $\psi: \mathcal{B} \rightarrow \mathcal{C}$ is a homomorphism to a finite algebra Ω -algebra $\mathcal{C} = (C, \Omega)$ and $H \subseteq C$. Then $L\varphi^{-1} = H(\varphi\psi)^{-1} \in \text{Rec}(\mathcal{A})$ as claimed. \square

Let us clarify here the relationship between the two notions of recognizable subsets, recognizable sorted subsets and pure recognizable subsets.

The following fact can be derived directly from Definition 3.1.

LEMMA 3.4 *Let $\mathcal{A} = (A, \Omega)$ be an S -sorted algebra. For any $s \in S$ and $T \subseteq A_s$, $T \in \text{Rec}(\mathcal{A}, s)$ iff $\langle T \rangle \in \text{Rec}(\mathcal{A})$. \square*

The forward direction of the following proposition is again a direct consequence of Definition 3.1, and the converse part follows from Lemma 3.4 and Proposition 3.3(2).

PROPOSITION 3.5 *A sorted subset $L = \langle L_s \rangle_{s \in S}$ of an S -sorted algebra $\mathcal{A} = (A, \Omega)$ is recognizable iff $L_s \in \text{Rec}(\mathcal{A}, s)$ for every $s \in S$. \square*

4 Syntactic congruences and algebras

We shall now present a theory of syntactic congruences and syntactic algebras for S -sorted algebras similar to those known for semigroups, monoids (cf. [8, 17, 18]) or general one-sorted algebras (cf. [1, 21, 22]).

DEFINITION 4.1 The *syntactic congruence* $\approx^L = \langle \approx_s^L \rangle_{s \in S}$ of a sorted subset L of an Ω -algebra $\mathcal{A} = (A, \Omega)$ is defined by

$$a \approx_s^L b \Leftrightarrow (\forall s' \in S)(\forall \alpha \in \text{Tr}(\mathcal{A}, s, s'))(\alpha(a) \in L_{s'} \leftrightarrow \alpha(b) \in L_{s'})$$

for every $s \in S$ and $a, b \in A_s$.

The following basic property of syntactic congruences can be verified exactly as in the one-sorted case.

LEMMA 4.2 *The syntactic congruence \approx^L of any sorted subset L of an Ω -algebra $\mathcal{A} = (A, \Omega)$ is the greatest congruence on \mathcal{A} that saturates L . \square*

Of course, we have also the following Nerode–Myhill type theorem.

PROPOSITION 4.3 *For any sorted subset L of an Ω -algebra $\mathcal{A} = (A, \Omega)$, the following are equivalent:*

- (1) $L \in \text{Rec}(\mathcal{A})$;
- (2) L is saturated by a congruence on \mathcal{A} of finite index;
- (3) \approx^L is of finite index.

Proof. If there exist a finite Ω -algebra $\mathcal{B} = (B, \Omega)$, a homomorphism $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ and a sorted subset H of B such that $L = H\varphi^{-1}$, then $\ker \varphi$ is a congruence on \mathcal{A} of finite index saturating L . On the other hand, if L is saturated by a congruence $\theta \in \text{Con}(\mathcal{A})$ of finite index, then L is recognized by the finite Ω -algebra \mathcal{A}/θ . Hence, (1) and (2) are equivalent. Conditions (2) and (3) are equivalent by Lemma 4.2. \square

Also the following facts can be proved similarly as their counterparts in the one-sorted theory. In the proposition, K and L are always sorted subsets.

PROPOSITION 4.4 *Let $\mathcal{A} = (A, \Omega)$ and $\mathcal{B} = (B, \Omega)$ be Ω -algebras.*

- (1) $\approx^{A-L} = \approx^L$, for every $L \subseteq A$.
- (2) $\approx^K \cap \approx^L \subseteq \approx^{K \cap L}$, for every $K, L \subseteq A$.
- (3) $\approx^L \subseteq \approx^{\alpha^{-1}(L)}$, for every $L \subseteq A$ and any translation $\alpha(\xi_s) \in \text{Tr}(\mathcal{A}, s, s')$.
- (4) If $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ is a homomorphism, then $\varphi \circ \approx^L \circ \varphi^{-1} \subseteq \approx^{L\varphi^{-1}}$ for every $L \subseteq B$. If φ is an epimorphism, then $\varphi \circ \approx^L \circ \varphi^{-1} = \approx^{L\varphi^{-1}}$. \square

For any sorted subset L of an Ω -algebra $\mathcal{A} = (A, \Omega)$, let $A/L = \langle A_s/L \rangle_{s \in S}$, where $A_s/L = A_s/\approx_s^L$ for each sort $s \in S$. Moreover, for any $s \in S$ and any $a \in A_s$, let a/L be a shorthand for a/\approx_s^L .

DEFINITION 4.5 The *syntactic algebra* $\mathcal{A}/L = (A/L, \Omega)$ of a sorted subset L of an Ω -algebra $\mathcal{A} = (A, \Omega)$ is the quotient algebra \mathcal{A}/\approx^L , and the corresponding canonical homomorphism $\varphi^L = \langle \varphi_s^L \rangle_{s \in S}$, where for each $s \in S$,

$$\varphi_s^L: A_s \rightarrow A_s/L, a \mapsto a/L, \quad (a \in A_s),$$

is called the *syntactic homomorphism* of L .

It is clear that any sorted subset L of an Ω -algebra $\mathcal{A} = (A, \Omega)$ is recognized by its syntactic algebra. Indeed, $L = L\varphi^L(\varphi^L)^{-1}$ for the syntactic homomorphism $\varphi^L: \mathcal{A} \rightarrow \mathcal{A}/L$. It follows from Lemma 4.2 that \mathcal{A}/L is in the following sense the least algebra recognizing L .

LEMMA 4.6 A sorted subset L of an Ω -algebra \mathcal{A} is recognizable iff the syntactic algebra \mathcal{A}/L is finite. An Ω -algebra \mathcal{B} recognizes L iff $\mathcal{A}/L \preceq \mathcal{B}$. \square

PROPOSITION 4.7 Let $\mathcal{A} = (A, \Omega)$ and $\mathcal{B} = (B, \Omega)$ be any Ω -algebras.

- (1) $\mathcal{A}/(A - L) = \mathcal{A}/L$, for any $L \subseteq A$.
- (2) $\mathcal{A}/K \cap L \preceq \mathcal{A}/K \times \mathcal{A}/L$, for any $K, L \subseteq A$.
- (3) $\mathcal{A}/\alpha^{-1}(L) \preceq \mathcal{A}/L$, for any $L \subseteq A$, $s, s' \in S$ and $\alpha(\xi_s) \in \text{Tr}(\mathcal{A}, s, s')$.
- (4) $\mathcal{A}/L\varphi^{-1} \preceq \mathcal{B}/L$, for any homomorphism $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ and any $L \subseteq B$.
Moreover, if φ is an epimorphism, then $\mathcal{A}/L\varphi^{-1} \cong \mathcal{B}/L$.

Proof. Assertions (1) and (3) follow immediately by the corresponding parts of Proposition 4.4 and Lemma 2.4. For (2) it suffices to note that

$$\mathcal{A}/K \cap L \leftarrow \mathcal{A}/(\approx^K \cap \approx^L) \subseteq \mathcal{A}/K \times \mathcal{A}/L$$

by Proposition 4.4(2) and Lemma 2.4.

To prove (4), let us first assume that φ is an epimorphism and show that

$$\psi_s: A_s/L\varphi^{-1} \rightarrow B_s/L, a/L\varphi^{-1} \mapsto a\varphi_s/L, \quad (s \in S, a \in A_s)$$

defines an isomorphism $\psi = \langle \psi_s \rangle_{s \in S}$ between $\mathcal{A}/L\varphi^{-1}$ and \mathcal{B}/L . First we verify that ψ is well-defined and injective: for each $s \in S$ and any $a, a' \in A_s$,

$$\begin{aligned} (a/L)\psi_s = (a'/L)\psi_s &\Leftrightarrow a\varphi_s \approx_s^L a'\varphi_s \\ &\Leftrightarrow (\forall s')(\forall \beta)[\beta(a\varphi_s) \in L_{s'} \leftrightarrow \beta(a'\varphi_s) \in L_{s'}] \\ &\Leftrightarrow (\forall s')(\forall \alpha)[\alpha_\varphi(a\varphi_s) \in L_{s'} \leftrightarrow \alpha_\varphi(a'\varphi_s) \in L_{s'}] \\ &\Leftrightarrow (\forall s')(\forall \alpha)[\alpha(a)\varphi_{s'} \in L_{s'} \leftrightarrow \alpha(a')\varphi_{s'} \in L_{s'}] \\ &\Leftrightarrow (\forall s')(\forall \alpha)[\alpha(a) \in L_{s'}\varphi_{s'}^{-1} \leftrightarrow \alpha(a') \in L_{s'}\varphi_{s'}^{-1}] \\ &\Leftrightarrow a/L\varphi^{-1} = a'/L\varphi^{-1}, \end{aligned}$$

where s' ranges over S , α over $\text{Tr}(\mathcal{A}, s, s')$ and β over $\text{Tr}(\mathcal{B}, s, s')$.

Consider now a homomorphism $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ that is not necessarily onto, and let $\mathcal{C} = (\langle A_s \varphi_s \varphi_s^L \rangle_{s \in S}, \Omega)$ be the subalgebra of \mathcal{B}/L obtained as the image of \mathcal{B} under the homomorphism $\varphi \varphi^L: \mathcal{A} \rightarrow \mathcal{B}/L$. Then $\eta: \mathcal{A} \rightarrow \mathcal{C}, a \mapsto a \varphi \varphi^L$, is an epimorphism, and hence $\mathcal{A}/L \varphi^{-1} \eta \eta^{-1} \cong \mathcal{C}/L \varphi^{-1} \eta$. However, this implies $\mathcal{A}/L \varphi^{-1} \preceq \mathcal{B}/L$ since $L \varphi^{-1} \eta \eta^{-1} = L \varphi^{-1}$ and \mathcal{C} is a subalgebra of \mathcal{B}/L . \square

LEMMA 4.8 *If $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ is a homomorphism of Ω -algebras and $L \subseteq B$, then for every $s \in S$,*

$$\varphi_s \circ \approx_s^L \circ \varphi_s^{-1} \subseteq \bigcap \{ \approx_s^{\beta^{-1}(L) \varphi^{-1}} \mid \beta \in \text{Tr}(\mathcal{B}, s, s'), s' \in S \},$$

and if φ is an epimorphism, equality holds.

Proof. Let ρ denote the intersection appearing in the claimed equality. Parts (3) and (4) of 4.4 yield for every $\beta \in \text{Tr}(\mathcal{B}, s, s')$,

$$\varphi_s \circ \approx_s^L \circ \varphi_s^{-1} \subseteq \varphi_s \circ \approx_s^{\beta^{-1}(L)} \circ \varphi_s^{-1} \subseteq \approx_s^{\beta^{-1}(L) \varphi^{-1}}.$$

Hence $\varphi_s \circ \approx_s^L \circ \varphi_s^{-1} \subseteq \rho$. Assume now that φ is surjective. The converse inclusion is then obtained by the following chain of implications, where $a, a' \in A_s$, s' and s'' range over S , β and γ are translations of \mathcal{B} , and $(\forall \beta)_{s, s'}$ is a shorthand for $(\forall \beta \in \text{Tr}(\mathcal{B}, s, s'))$ etc.:

$$\begin{aligned} a \rho a' &\Rightarrow (\forall s') (\forall \beta)_{s, s'} [a \approx_s^{\beta^{-1}(L) \varphi^{-1}} a'] \\ &\Rightarrow (\forall s') (\forall \beta)_{s, s'} [a \varphi_s \approx_s^{\beta^{-1}(L)} a' \varphi_s] \\ &\Rightarrow (\forall s', s'') (\forall \beta)_{s, s'} (\forall \gamma)_{s, s''} [\gamma(a \varphi_s) \in \beta^{-1}(L)_{s''} \leftrightarrow \gamma(a' \varphi_s) \in \beta^{-1}(L)_{s''}] \\ &\Rightarrow (\forall s') (\forall \beta)_{s, s'} (\forall \gamma)_{s, s} [\gamma(a \varphi_s) \in \beta^{-1}(L)_{s'} \leftrightarrow \gamma(a' \varphi_s) \in \beta^{-1}(L)_{s'}] \\ &\Rightarrow (\forall s') (\forall \beta)_{s, s'} (\forall \gamma)_{s, s} [\beta(\gamma(a \varphi_s)) \in L_{s'} \leftrightarrow \beta(\gamma(a' \varphi_s)) \in L_{s'}] \\ &\Rightarrow (\forall s') (\forall \beta)_{s, s'} [\beta(a \varphi_s) \in L_{s'} \leftrightarrow \beta(a' \varphi_s) \in L_{s'}] \\ &\Rightarrow a \varphi_s \approx_s^L a' \varphi_s \\ &\Rightarrow a \varphi_s \circ \approx_s^L \circ \varphi_s^{-1} a'. \end{aligned}$$

Here we used also the fact that $\beta^{-1}(L)_{s''} = \emptyset$ for every $s'' \neq s$. \square

Let us now present the natural generalizations of some basic facts known for monoids [8, 17] and algebras in general in the one-sorted case [21, 22].

LEMMA 4.9 *Let $L = \langle L_s \rangle_{s \in S}$ be a sorted subset of an Ω -algebra $\mathcal{A} = (A, \Omega)$. For any $s \in S$ and $a \in A_s$,*

$$a/L = \bigcap \{ \alpha^{-1}(L_{s'}) \mid \alpha(a_s) \in L_{s'} \} - \bigcup \{ \alpha^{-1}(L_{s'}) \mid \alpha(a_s) \notin L_{s'} \},$$

where s' ranges over S and α over $\text{Tr}(\mathcal{A}, s, s')$. \square

LEMMA 4.10 *Any congruence θ on an algebra $\mathcal{A} = (A, \Omega)$ is the intersection of some syntactic congruences. In particular, $\theta = \bigcap \{\approx^{(a/\theta)} \mid s \in S, a \in A_s\}$. \square*

Let us call an Ω -algebra \mathcal{A} *syntactic*, if $\mathcal{A} \cong \mathcal{B}/L$ for some Ω -algebra \mathcal{B} and some sorted subset L of \mathcal{B} . A sorted subset D of an Ω -algebra \mathcal{A} is *disjunctive* if $\approx^D = \Delta_{\mathcal{A}}$.

PROPOSITION 4.11 *An Ω -algebra \mathcal{A} is syntactic iff it has a disjunctive subset. \square*

Subdirect products of Ω -algebras are defined (cf. [15], Section 4.1, or [13], p. 159) exactly as for one-sorted algebras, and by generalizing in an obvious way a well-known theorem of Birkhoff (cf. [3], for example), we may say that an Ω -algebra $\mathcal{A} = (A, \Omega)$ is *subdirectly irreducible* if the intersection of all non-trivial congruences on \mathcal{A} is the diagonal relation $\Delta_{\mathcal{A}}$. By applying Lemma 4.10 to the diagonal relation we get the following result.

COROLLARY 4.12 *Every subdirectly irreducible Ω -algebra is syntactic. \square*

Since it is clear that also varieties of many-sorted algebras are generated by their subdirectly irreducible members, Corollary 4.12 implies the following important fact. However, let us note that the result follows also directly from Lemma 4.10: we have $\mathcal{A} \subseteq \prod \{\mathcal{A}/\approx^{\{a\}} \mid a \in A\}$ for any finite $\mathcal{A} = (A, \Omega)$ since $\Delta_{\mathcal{A}} = \bigcap \{\approx^{\{a\}} \mid a \in A\}$.

LEMMA 4.13 *Every Ω -VFA is generated by syntactic algebras. Hence, if \mathbf{K} is an Ω -VFA and \mathcal{A} any finite Ω -algebra, then $\mathcal{A} \in \mathbf{K}$ iff $\mathcal{A} \preceq \mathcal{A}_1 \times \dots \times \mathcal{A}_n$ for some $n \geq 0$ and some syntactic algebras $\mathcal{A}_1, \dots, \mathcal{A}_n \in \mathbf{K}$. \square*

5 Varieties of recognizable V-sets and V-congruences

Let S and Ω be again a finite set of sorts and an S -sorted ranked alphabet, respectively. We shall consider varieties of recognizable subsets of finitely generated free algebras over a given variety \mathbf{V} of Ω -algebras. If \mathbf{V} is the class of all Ω -algebras, we are actually dealing with varieties of many-sorted tree languages. In what follows, we call finite S -sorted alphabets full for Ω simply *full alphabets*, and $X = \langle X_s \rangle_{s \in S}$ and $Y = \langle Y_s \rangle_{s \in S}$ are always such full alphabets.

The free algebra $\mathcal{F}_{\mathbf{V}}(X) = (\mathbf{F}_{\mathbf{V}}(X), \Omega)$ exists for every full alphabet X , and we call the recognizable subsets of $\mathcal{F}_{\mathbf{V}}(X)$ *recognizable V-sets*. The syntactic algebra $\mathcal{F}_{\mathbf{V}}(X)/L$ of a sorted subset L of $\mathcal{F}_{\mathbf{V}}(X)$ is denoted simply by $\text{SA}(L)$. It is clear that $\text{SA}(L) \in \mathbf{V}$.

We shall also need the following fact that can be proved similarly as its one-sorted counterpart [8, 21, 22].

LEMMA 5.1 *Let \mathcal{A} be a finite algebra in \mathbf{V} and let X be a full alphabet such that for some generating set $G = \langle G_s \rangle_{s \in S}$ of \mathcal{A} , $|G_s| \leq |X_s|$ for every $s \in S$. Then \mathcal{A} is syntactic iff $\mathcal{A} \cong \text{SA}(L)$ for some $L \in \text{Rec}(\mathcal{F}_{\mathbf{V}}(X))$. \square*

A family of recognizable \mathbf{V} -sets is a mapping \mathcal{R} that assigns to each full alphabet X a set $\mathcal{R}(X) \subseteq \text{Rec}(\mathcal{F}_{\mathbf{V}}(X))$ of recognizable \mathbf{V} -sets. We write then $\mathcal{R} = \{\mathcal{R}(X)\}_X$ with the understanding that X ranges over all full alphabets. The inclusion relation and the basic set-operations are defined for families of recognizable \mathbf{V} -sets by the natural componentwise conditions. For example, if \mathcal{R}_1 and \mathcal{R}_2 are any families of recognizable \mathbf{V} -sets, then $\mathcal{R}_1 \subseteq \mathcal{R}_2$ iff $\mathcal{R}_1(X) \subseteq \mathcal{R}_2(X)$ for every X .

For any X and $L \subseteq \mathbf{F}_{\mathbf{V}}(X)$, let \bar{L} denote the complement $\mathbf{F}_{\mathbf{V}}(X) - L$.

DEFINITION 5.2 A family of recognizable \mathbf{V} -sets $\mathcal{R} = \{\mathcal{R}(X)\}_X$ is a *variety of recognizable \mathbf{V} -sets*, a \mathbf{V} -VRS for short, if for all full alphabets X and Y ,

- (1) $\mathcal{R}(X) \neq \emptyset$,
- (2) $K, L \in \mathcal{R}(X)$ implies $K \cap L, \bar{L} \in \mathcal{R}(X)$,
- (3) if $L \in \mathcal{R}(X)$, then $\alpha^{-1}(L) \in \mathcal{R}(X)$ for every $\alpha \in \text{Tr}(\mathcal{F}_{\mathbf{V}}(X))$, and
- (4) if $L \in \mathcal{R}(Y)$, then $L\varphi^{-1} \in \mathcal{R}(X)$ for every $\varphi : \mathcal{F}_{\mathbf{V}}(X) \rightarrow \mathcal{F}_{\mathbf{V}}(Y)$.

Let $\text{VRS}(\mathbf{V})$ denote the class of all varieties of recognizable \mathbf{V} -sets.

It is clear that the intersection of any family of varieties of recognizable \mathbf{V} -sets is again a \mathbf{V} -VRS, and hence $(\text{VRS}(\mathbf{V}), \subseteq)$ is a complete (in fact, algebraic) lattice.

If $L = \langle L_s \rangle_{s \in S}$ is a sorted subset of any algebra $\mathcal{A} = (A, \Omega)$ and $s \in S$ is any sort, then $\langle L_s \rangle = 1_{A(s)}^{-1}(L)$. Applied to the algebras $\mathcal{F}_{\mathbf{V}}(X)$, this observation yields the following fact.

LEMMA 5.3 *Let $\mathcal{R} = \{\mathcal{R}(X)\}_X$ be a \mathbf{V} -VRS. If $L = \langle L_s \rangle_{s \in S} \in \mathcal{R}(X)$ for some X , then $\langle L_s \rangle \in \mathcal{R}(X)$ for every $s \in S$. \square*

From Lemma 5.3 and Lemma 4.9 we get directly the following fact.

LEMMA 5.4 *If $\mathcal{R} = \{\mathcal{R}(X)\}_X$ is a \mathbf{V} -VRS and $L \in \mathcal{R}(X)$ for some X , then $\langle a/L \rangle$ belongs to $\mathcal{R}(X)$ for any $s \in S$ and any $a \in \mathbf{F}_{\mathbf{V}}(X, s)$. \square*

For any full alphabet X , let $\text{FCon}(\mathcal{F}_{\mathbf{V}}(X))$ denote the set of all congruences on $\mathcal{F}_{\mathbf{V}}(X)$ of finite index. These congruences are called *\mathbf{V} -congruences*, and a family of \mathbf{V} -congruences is a map Γ that assigns to each X a set $\Gamma(X) \subseteq \text{FCon}(\mathcal{F}_{\mathbf{V}}(X))$. We represent such a family in the form $\Gamma = \{\Gamma(X)\}_X$.

DEFINITION 5.5 A family of \mathbf{V} -congruences $\Gamma = \{\Gamma(X)\}_X$ is said to be a *variety of \mathbf{V} -congruences*, a \mathbf{V} -VFC for short, if for all X and Y ,

- (1) $\Gamma(X) \neq \emptyset$,
- (2) if $\theta, \theta' \in \Gamma(X)$, then $\theta \cap \theta' \in \Gamma(X)$,
- (3) if $\theta \in \Gamma(X)$ and $\theta \subseteq \theta'$, then $\theta' \in \Gamma(X)$, and
- (4) if $\theta \in \Gamma(Y)$ and $\varphi : \mathcal{F}_{\mathbf{V}}(X) \rightarrow \mathcal{F}_{\mathbf{V}}(Y)$, then $\varphi \circ \theta \circ \varphi^{-1} \in \Gamma(X)$.

Let $\text{VFC}(\mathbf{V})$ denote the class of all varieties of \mathbf{V} -congruences.

In other words, a variety of \mathbf{V} -congruences is a family of filters of the lattices $\text{FCon}(\mathcal{F}_{\mathbf{V}}(X))$ closed under inverse homomorphisms. It is again easy to see that $(\text{VFC}(\mathbf{V}), \subseteq)$ is an algebraic lattice.

6 The Variety Theorem

Let S, Ω and \mathbf{V} be as in the previous section. By a *variety of finite \mathbf{V} -algebras*, a \mathbf{V} -VFA for short, we mean a variety of finite Ω -algebras contained in \mathbf{V} . Let $\text{VFA}(\mathbf{V})$ be the class of all \mathbf{V} -VFAs. We shall prove a Variety Theorem that establishes a triple of bijective correspondences between all varieties of recognizable \mathbf{V} -sets, all varieties of finite \mathbf{V} -algebras, and all varieties of \mathbf{V} -congruences. The proof is similar to those of various other Variety Theorems, and in particular to the one of [22]. However, for the convenience of the reader, the following presentation is self-contained.

Let us now introduce the six mappings that will yield the Variety Theorem in the form of three pairs of mutually inverse isomorphisms between the three complete lattices $(\text{VFA}(\mathbf{V}), \subseteq)$, $(\text{VRS}(\mathbf{V}), \subseteq)$ and $(\text{VFC}(\mathbf{V}), \subseteq)$.

DEFINITION 6.1 For any \mathbf{V} -VFA \mathbf{K} , any \mathbf{V} -VRS \mathcal{R} , and any \mathbf{V} -VFC Γ , let

- (1) \mathbf{K}^r be the family of recognizable \mathbf{V} -sets such that for each X ,

$$\mathbf{K}^r(X) = \{L \subseteq \text{F}_{\mathbf{V}}(X) \mid \text{SA}(L) \in \mathbf{K}\},$$

- (2) \mathbf{K}^c be the family of \mathbf{V} -congruences such that for each X ,

$$\mathbf{K}^c(X) = \{\theta \in \text{FCon}(\mathcal{F}_{\mathbf{V}}(X)) \mid \mathcal{F}_{\mathbf{V}}(X)/\theta \in \mathbf{K}\},$$

- (3) \mathcal{R}^a be the \mathbf{V} -VFA generated by the syntactic algebras $\text{SA}(L)$ with $L \in \mathcal{R}(X)$ for some X ,
- (4) \mathcal{R}^c be the family of \mathbf{V} -congruences such that for each X , $\mathcal{R}^c(X)$ is the filter in the lattice $\text{FCon}(\mathcal{F}_{\mathbf{V}}(X))$ generated by the syntactic congruences \approx^L of all sets $L \in \mathcal{R}(X)$,
- (5) Γ^a be the \mathbf{V} -VFA generated by all algebras $\mathcal{F}_{\mathbf{V}}(X)/\theta$ such that $\theta \in \mathcal{R}(X)$ for some X , and let

(6) Γ^r be the family of recognizable \mathbf{V} -sets such that for each X ,

$$\Gamma^r(X) = \{L \subseteq F_{\mathbf{V}}(X) \mid \approx^L \in \Gamma(X)\}.$$

LEMMA 6.2 *For any $\mathbf{K} \in \text{VFA}(\mathbf{V})$, $\mathcal{R} \in \text{VRS}(\mathbf{V})$ and $\Gamma \in \text{VFC}(\mathbf{V})$,*

- (1) $\mathcal{R}^a, \Gamma^a \in \text{VFA}(\mathbf{V})$,
- (2) $\mathbf{K}^r, \Gamma^r \in \text{VRS}(\mathbf{V})$, and
- (3) $\mathbf{K}^c, \mathcal{R}^c \in \text{VFC}(\mathbf{V})$.

Moreover, all of the mappings $\mathbf{K} \mapsto \mathbf{K}^r$, $\mathbf{K} \mapsto \mathbf{K}^c$, $\mathcal{R} \mapsto \mathcal{R}^a$, $\mathcal{R} \mapsto \mathcal{R}^c$, $\Gamma \mapsto \Gamma^a$ and $\Gamma \mapsto \Gamma^r$ are inclusion-preserving.

Proof. By definition, \mathcal{R}^a and Γ^a are \mathbf{V} -VFAs. That \mathbf{K}^r and Γ^r are \mathbf{V} -VRSs, follows from Propositions 4.7 and 4.4. Finally, Lemmas 2.4 and 4.4, and Proposition 4.4 imply that \mathbf{K}^c and \mathcal{R}^c are in $\text{VFC}(\mathbf{V})$. \square

We shall show that the six mappings introduced above form three pairs of mutually inverse isomorphisms between the complete lattices $(\text{VFA}(\mathbf{V}), \subseteq)$, $(\text{VRS}(\mathbf{V}), \subseteq)$ and $(\text{VFC}(\mathbf{V}), \subseteq)$. Since we already know that all of the maps are isotone, it suffices to show that they are pairwise inverses of each other.

PROPOSITION 6.3 *The lattices $(\text{VFA}(\mathbf{V}), \subseteq)$ and $(\text{VRS}(\mathbf{V}), \subseteq)$ are isomorphic as*

- (1) $\mathbf{K}^{ra} = \mathbf{K}$ for every $\mathbf{K} \in \text{VFA}(\mathbf{V})$, and
- (2) $\mathcal{R}^{ar} = \mathcal{R}$ for every $\mathcal{R} \in \text{VRS}(\mathbf{V})$.

Proof. It suffices to prove (1) and (2).

Since \mathbf{K}^{ra} is generated by syntactic algebras belonging to \mathbf{K} , the inclusion $\mathbf{K}^{ra} \subseteq \mathbf{K}$ is obvious. For the converse inclusion, let us consider any syntactic algebra $\mathcal{A} \in \mathbf{K}$. By Lemma 5.1 there exists an X such that $\mathcal{A} \cong \text{SA}(L)$ for some $L \in \text{Rec}(\mathcal{F}_{\mathbf{V}}(X))$. Then $L \in \mathbf{K}^r(X)$ and hence $\mathcal{A} \in \mathbf{K}^{ra}$. This implies $\mathbf{K} \subseteq \mathbf{K}^{ra}$ because, by Lemma 4.13, \mathbf{K} is generated by syntactic algebras.

The inclusion $\mathcal{R} \subseteq \mathcal{R}^{ar}$ is obvious: if $L \in \mathcal{R}(X)$ for any X , then $\text{SA}(L) \in \mathcal{R}^a$ and hence $L \in \mathcal{R}^{ar}(X)$. Assume then that $L \in \mathcal{R}^{ar}(X)$ for some X . Then $\text{SA}(L) \in \mathcal{R}^a$ implies that $\text{SA}(L) \preceq \text{SA}(L_1) \times \cdots \times \text{SA}(L_k)$ for some $k \geq 1$, some full alphabets $X_i = \langle X_i(s) \rangle_{s \in S}$ and sets $L_i \in \mathcal{R}(X_i)$ ($i = 1, \dots, k$). For each $i = 1, \dots, k$, let φ_i denote the syntactic homomorphisms $\varphi^{L_i}: \mathcal{F}_{\mathbf{V}}(X_i) \rightarrow \text{SA}(L_i)$. Then there is a homomorphism

$$\eta: \mathcal{F}_{\mathbf{V}}(X_1) \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k) \longrightarrow \text{SA}(L_1) \times \cdots \times \text{SA}(L_k)$$

such that for every $i = 1, \dots, k$, $\eta\pi_i = \tau_i\varphi_i$, where

$$\pi_i: \text{SA}(L_1) \times \cdots \times \text{SA}(L_k) \longrightarrow \text{SA}(L_i),$$

and

$$\tau_i: F_{\mathbf{V}}(X_1) \times \cdots \times F_{\mathbf{V}}(X_k) \longrightarrow F_{\mathbf{V}}(X_i)$$

are projection functions. By Lemma 4.6 there exist a homomorphism $\varphi: \mathcal{F}_{\mathbf{V}}(X) \rightarrow \text{SA}(L_1) \times \cdots \times \text{SA}(L_k)$ and a subset H of the product $\text{SA}(L_1) \times \cdots \times \text{SA}(L_k)$ such that $L = H\varphi^{-1}$. Since η is an epimorphism, there is a homomorphism $\psi: \mathcal{F}_{\mathbf{V}}(X) \rightarrow \mathcal{F}_{\mathbf{V}}(X_1) \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k)$ such that $\psi\eta = \varphi$. Because H is finite, $L = \bigcup_{u \in H} u\varphi^{-1}$ is the union of finitely many sets $u\varphi^{-1}$ with $u = (u_1, \dots, u_k) \in \text{SA}(L_1) \times \cdots \times \text{SA}(L_k)$. For each such $u \in H$,

$$u\varphi^{-1} = \bigcap \{u_i(\varphi\pi_i)^{-1} \mid 1 \leq i \leq k\} = \bigcap \{u_i\varphi_i^{-1}(\psi\tau_i)^{-1} \mid 1 \leq i \leq k\}.$$

By Lemma 5.4, $u_i\varphi_i^{-1} \in \mathcal{R}(X_i)$ for each $i = 1, \dots, k$, and thus $L \in \mathcal{R}(X)$. \square

LEMMA 6.4 *For any \mathbf{V} -VFC Γ and any finite algebra $\mathcal{A} \in \mathbf{V}$, $\mathcal{A} \in \Gamma^a$ iff there exist an X and an epimorphism $\varphi: \mathcal{F}_{\mathbf{V}}(X) \rightarrow \mathcal{A}$ such that $\ker \varphi \in \Gamma(X)$.*

Proof. If $\mathcal{A} \in \Gamma^a$, then $\mathcal{A} \preceq \mathcal{F}_{\mathbf{V}}(X_1)/\theta_1 \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k)/\theta_k$ for some $k \geq 1$, some full alphabets X_1, \dots, X_k and congruences $\theta_1 \in \Gamma(X_1), \dots, \theta_k \in \Gamma(X_k)$. This means that for some algebra \mathcal{B} there exist an epimorphism $\eta: \mathcal{B} \rightarrow \mathcal{A}$ and a monomorphism $\varphi: \mathcal{B} \rightarrow \mathcal{F}_{\mathbf{V}}(X_1)/\theta_1 \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k)/\theta_k$. The algebras $\mathcal{F}_{\mathbf{V}}(X_i)/\theta_i$ are finite members of \mathbf{V} and hence there is for some X an epimorphism $\psi: \mathcal{F}_{\mathbf{V}}(X) \rightarrow \mathcal{B}$. The condition $(a_1, \dots, a_k)\chi = (a_1/\theta_1, \dots, a_k/\theta_k)$ defines an epimorphism

$$\chi: \mathcal{F}_{\mathbf{V}}(X_1) \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k) \longrightarrow \mathcal{F}_{\mathbf{V}}(X_1)/\theta_1 \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k)/\theta_k.$$

For each $i = 1, \dots, k$, let $\pi_i: \mathcal{F}_{\mathbf{V}}(X_1) \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k) \rightarrow \mathcal{F}_{\mathbf{V}}(X_i)$ be the i^{th} projection, and let $\omega: \mathcal{F}_{\mathbf{V}}(X) \rightarrow \mathcal{F}_{\mathbf{V}}(X_1) \times \cdots \times \mathcal{F}_{\mathbf{V}}(X_k)$ be the homomorphism such that $\omega\chi = \psi\varphi$. Then $\psi\eta: \mathcal{F}_{\mathbf{V}}(X) \rightarrow \mathcal{A}$ is an epimorphism, and

$$\ker \psi\eta \supseteq \ker \psi\varphi = \ker \omega\chi = \bigcap \{\omega\pi_i \circ \theta_i \circ (\omega\pi_i)^{-1} \mid 1 \leq i \leq k\}$$

shows that $\ker \psi\eta \in \Gamma(X)$.

The converse implication is immediately clear by the definition of Γ^a . \square

PROPOSITION 6.5 *The lattices $(\text{VFA}(\mathbf{V}), \subseteq)$ and $(\text{VFC}(\mathbf{V}), \subseteq)$ are isomorphic as*

- (1) $\mathbf{K}^{ca} = \mathbf{K}$ for every \mathbf{V} -VFA \mathbf{K} , and
- (2) $\Gamma^{ac} = \Gamma$ for every \mathbf{V} -VFC Γ .

Proof. By Lemma 6.4, $\mathcal{A} \in \mathbf{K}^{ca}$ holds iff for some X there exists an epimorphism $\varphi: \mathcal{F}_{\mathbf{V}}(X) \rightarrow \mathcal{A}$ such that $\ker \varphi \in \mathbf{K}^c$. By Proposition 2.1 this is equivalent to $\mathcal{F}_{\mathbf{V}}(X)/\ker \varphi \cong \mathcal{A}$, which is the case exactly when $\mathcal{A} \in \mathbf{K}$. Thus (1) follows.

To prove (2), consider any X and any $\theta \in \text{FCon}(\mathcal{F}_{\mathbf{V}}(X))$. If $\theta \in \Gamma^{ac}(X)$, then by Lemma 6.4, there exist a finite set Y and an epimorphism $\psi: \mathcal{F}_{\mathbf{V}}(Y) \rightarrow$

$\mathcal{F}_{\mathbf{V}}(X)/\theta$ such that $\ker \psi \in \Gamma(Y)$. Since ψ is surjective, there is for any $s \in S$ and every $x \in X_s$ an element $t_s^x \in F_{\mathbf{V}}(Y)_s$ such that $t_s^x \psi_s = x/\theta_s$. If $\varphi: \mathcal{F}_{\mathbf{V}}(X) \rightarrow \mathcal{F}_{\mathbf{V}}(Y)$ is the homomorphism such that $x\varphi = t_s^x$ for all $s \in S$ and $x \in X_s$, then $\varphi\psi = \theta^{\natural}$, where $\theta^{\natural}: \mathcal{F}_{\mathbf{V}}(X) \rightarrow \mathcal{F}_{\mathbf{V}}(X)/\theta$ is the canonical epimorphism. Hence $\theta = \ker \varphi\psi = \varphi \circ (\ker \psi) \circ \varphi^{-1} \in \Gamma(X)$. The converse inclusion is obvious: if $\theta \in \Gamma(X)$, then $\mathcal{F}_{\mathbf{V}}(X)/\theta \in \Gamma^a$ implies $\theta \in \Gamma^{ac}$. \square

Propositions 6.3 and 6.5 already show that $(\text{VRS}(\mathbf{V}), \subseteq)$ and $(\text{VFC}(\mathbf{V}), \subseteq)$ are isomorphic lattices, but the following composition laws imply also that the mappings $\mathcal{R} \mapsto \mathcal{R}^c$ and $\Gamma \mapsto \Gamma^r$ form a pair of mutually inverse isomorphisms between them.

PROPOSITION 6.6 *For any \mathbf{V} -VFA \mathbf{K} , \mathbf{V} -VRS \mathcal{R} , and \mathbf{V} -VFC Γ ,*

- (1) $\mathbf{K}^{cr} = \mathbf{K}^r$,
- (2) $\mathcal{R}^{ac} = \mathcal{R}^c$, and
- (3) $\Gamma^{ra} = \Gamma^a$.

Proof. For (1) it suffices to note that

$$L \in \mathbf{K}^r(X) \Leftrightarrow \text{SA}(L) \in \mathbf{K} \Leftrightarrow \approx^L \in \mathbf{K}^c(X) \Leftrightarrow L \in \mathbf{K}^{cr}(X),$$

for any X and $L \subseteq F_{\mathbf{V}}(X)$.

To prove (2), let us consider any X and $\text{FCon}(\mathcal{F}_{\mathbf{V}}(X))$. If $\theta \in \mathcal{R}^c(X)$, then $\approx^{L_1} \cap \dots \cap \approx^{L_k} \subseteq \theta$ for some $k \geq 1$ and $L_1, \dots, L_k \in \mathcal{R}(X)$. This implies that $\mathcal{F}_{\mathbf{V}}(X)/\theta \in \mathcal{R}^a$ since $\mathcal{F}_{\mathbf{V}}(X)/\theta \preceq \text{SA}(L_1) \times \dots \times \text{SA}(L_k)$, and therefore $\theta \in \mathcal{R}^{ac}$.

If $\theta \in \mathcal{R}^{ac}(X)$, then $\mathcal{F}_{\mathbf{V}}(X)/\theta \preceq \text{SA}(L_1) \times \dots \times \text{SA}(L_k)$ for some full alphabets X_1, \dots, X_k and sorted sets $L_1 \in \mathcal{R}(X_1), \dots, L_k \in \mathcal{R}(X_k)$ ($k \geq 1$). Hence, there is an Ω -algebra \mathcal{B} such that there exist an epimorphism $\psi: \mathcal{B} \rightarrow \mathcal{F}_{\mathbf{V}}(X)/\theta$ and a monomorphism $\eta: \mathcal{B} \rightarrow \text{SA}(L_1) \times \dots \times \text{SA}(L_k)$. We may also assume that there is an epimorphism $\varphi: \mathcal{F}_{\mathbf{V}}(X) \rightarrow \mathcal{B}$ such that $\varphi\psi = \theta^{\natural}$ (if not, we replace \mathcal{B} with a suitable subalgebra). For each $i = 1, \dots, k$, let π_i be the i^{th} projection from $\mathcal{F}_{\mathbf{V}}(X_1) \times \dots \times \mathcal{F}_{\mathbf{V}}(X_k)$ onto $\mathcal{F}_{\mathbf{V}}(X_i)$, and let

$$\pi: \mathcal{F}_{\mathbf{V}}(X_1) \times \dots \times \mathcal{F}_{\mathbf{V}}(X_k) \longrightarrow \text{SA}(L_1) \times \dots \times \text{SA}(L_k)$$

be the homomorphism such that $(t_1, \dots, t_k) \mapsto (t_1/L_1, \dots, t_k/L_k)$ for all $s \in S$ and $t_1 \in F_{\mathbf{V}}(X_1, s), \dots, t_k \in F_{\mathbf{V}}(X_k, s)$. Since π clearly is surjective, we may define a homomorphism $\gamma: \mathcal{F}_{\mathbf{V}}(X) \rightarrow \mathcal{F}_{\mathbf{V}}(X_1) \times \dots \times \mathcal{F}_{\mathbf{V}}(X_k)$ for which $\gamma\pi = \varphi\eta$. Then

$$\theta = \ker \varphi\psi \supseteq \ker \varphi\eta = \ker \gamma\pi = \bigcap \{ \gamma\pi_i \circ \approx^{L_i} \circ (\gamma\pi_i)^{-1} \mid 1 \leq i \leq k \},$$

and hence $\theta \in \mathcal{R}^c(X)$.

To prove (3), consider any finite algebra $\mathcal{A} = (A, \Omega)$. Now, \mathcal{A} belongs to Γ^a iff $\mathcal{A} \preceq \mathcal{F}_{\mathbf{V}}(X_1)/\theta_1 \times \dots \times \mathcal{F}_{\mathbf{V}}(X_k)/\theta_k$, for some full alphabets X_1, \dots, X_k

and some $\theta_1 \in \Gamma(X_1), \dots, \theta_k \in \Gamma(X_k)$ ($k \geq 1$). Since any $\Gamma(X)$ is generated by syntactic congruences by Lemma 4.10, we can assume that each θ_i is the syntactic congruence of some $L_i \subseteq F_{\mathbf{V}}(X_i)$, and then $L_i \in \Gamma^r(X_i)$, and so $\mathcal{A} \in \Gamma^a$ iff $\mathcal{A} \in \Gamma^{ra}$. \square

PROPOSITION 6.7 *The lattices $(\text{VRS}(\mathbf{V}), \subseteq)$ and $(\text{VFC}(\mathbf{V}), \subseteq)$ are isomorphic as*

- (1) $\mathcal{R}^{cr} = \mathcal{R}$ for every $\mathcal{R} \in \text{VRS}(\mathbf{V})$, and
- (2) $\Gamma^{rc} = \Gamma$ for every $\Gamma \in \text{VFC}(\mathbf{V})$.

Proof. By using the previous propositions we can see that $\mathcal{R}^{cr} = \mathcal{R}^{acr} = \mathcal{R}^{ar} = \mathcal{R}$ for every $\mathcal{R} \in \text{VRS}(\mathbf{V})$. Similarly, $\Gamma^{rc} = \Gamma^{rac} = \Gamma^{ac} = \Gamma$ for every $\Gamma \in \text{VFC}(\mathbf{V})$. \square

Let us note that Proposition 6.7 could be obtained also directly in a similar way as the analogous facts are proved in [1]. For example, $\mathcal{R}^{cr} = \mathcal{R}$ can be seen as follows.

The inclusion $\mathcal{R} \subseteq \mathcal{R}^{cr}$ follows directly from the definitions of the two operators. On the other hand, if $L \in \mathcal{R}^{cr}(X)$, then $\approx^{L_1} \cap \dots \cap \approx^{L_k} \subseteq \approx^L$ for some L_1, \dots, L_k in $\mathcal{R}(X)$. This means that each \approx^L -class, and hence also L , is a Boolean combination of \approx^{L_i} -classes ($1 \leq i \leq k$), and since each such class is in $\mathcal{R}(X)$ by Lemma 5.4, this implies $L \in \mathcal{R}(X)$.

We may sum up the results of this section as follows.

THEOREM 6.8 (VARIETY THEOREM) *The mappings*

$$\begin{aligned} \text{VFA}(\mathbf{V}) \rightarrow \text{VRS}(\mathbf{V}), \mathbf{K} \mapsto \mathbf{K}^r & \quad \text{VRS}(\mathbf{V}) \rightarrow \text{VFA}(\mathbf{V}), \mathcal{R} \mapsto \mathcal{R}^a \\ \text{VFA}(\mathbf{V}) \rightarrow \text{VFC}(\mathbf{V}), \mathbf{K} \mapsto \mathbf{K}^c & \quad \text{VFC}(\mathbf{V}) \rightarrow \text{VFA}(\mathbf{V}), \Gamma \mapsto \Gamma^a \\ \text{VRS}(\mathbf{V}) \rightarrow \text{VFC}(\mathbf{V}), \mathcal{R} \mapsto \mathcal{R}^c & \quad \text{VFC}(\mathbf{V}) \rightarrow \text{VRS}(\mathbf{V}), \Gamma \mapsto \Gamma^r \end{aligned}$$

form three pairs of isomorphisms that are inverses of each other between the complete lattices $(\text{VFA}(\mathbf{V}), \subseteq)$, $(\text{VRS}(\mathbf{V}), \subseteq)$, and $(\text{VFC}(\mathbf{V}), \subseteq)$. Moreover, $\mathbf{K}^{cr} = \mathbf{K}^r$, $\mathbf{K}^{rc} = \mathbf{K}^c$, $\mathcal{R}^{ca} = \mathcal{R}^a$, $\mathcal{R}^{ac} = \mathcal{R}^c$, $\Gamma^{ra} = \Gamma^a$, and $\Gamma^{ar} = \Gamma^r$, for any $\mathbf{K} \in \text{VFA}(\mathbf{V})$, $\mathcal{R} \in \text{VRS}(\mathbf{V})$, and $\Gamma \in \text{VFC}(\mathbf{V})$.

Proof. That the given mappings form isomorphisms of the claimed kind follows from Propositions 6.3, 6.5, and 6.7. Moreover, Proposition 6.6 contains half of the composition laws, and together with Propositions 6.3, 6.5 and 6.7 it implies also the rest of them. For example, by Propositions 6.6 and 6.3, we get $\mathbf{K}^{rc} = (\mathbf{K}^r)^{ac} = (\mathbf{K}^{ra})^c = \mathbf{K}^c$ for any $\mathbf{K} \in \text{VFA}(\mathbf{V})$. \square

7 Varieties of pure recognizable sets

In this section we shall show how the above variety theory can be translated into a theory of varieties of pure recognizable sets.

Let \mathbf{V} be again a given variety of Ω -algebras. For any full alphabet X and any sort $s \in S$, the members of $\text{Rec}(\mathcal{F}_{\mathbf{V}}(X), s)$ are called *pure recognizable $\mathbf{V}X$ -sets of sort s* , or simply *pure recognizable \mathbf{V} -sets*. A *family of pure recognizable \mathbf{V} -sets* is a mapping \mathcal{P} that assigns to each X and each $s \in S$ a set $\mathcal{P}(X, s) \subseteq \text{Rec}(\mathcal{F}_{\mathbf{V}}(X), s)$ of pure recognizable $\mathbf{V}X$ -sets of sort s , and we write it as $\mathcal{P} = \{\mathcal{P}(X, s)\}_{X, s}$.

DEFINITION 7.1 A *variety of pure recognizable \mathbf{V} -sets*, a \mathbf{V} -VPRS for short, is a family of pure recognizable \mathbf{V} -sets $\mathcal{P} = \{\mathcal{P}(X, s)\}_{X, s}$ such that for all full alphabets X and Y and all sorts $s, s' \in S$,

- (1) $\mathcal{P}(X, s) \neq \emptyset$,
- (2) $T, U \in \mathcal{P}(X, s)$ implies $T \cap U, \overline{T} \in \mathcal{P}(X, s)$,
- (3) if $T \in \mathcal{P}(X, s')$ and $\alpha \in \text{Tr}(\mathcal{F}_{\mathbf{V}}(X), s, s')$, then $\alpha^{-1}(T) \in \mathcal{P}(X, s)$, and
- (4) if $T \in \mathcal{P}(Y, s)$ and $\varphi: \mathcal{F}_{\mathbf{V}}(X) \rightarrow \mathcal{F}_{\mathbf{V}}(Y)$ is any homomorphism, then $T\varphi_s^{-1} \in \mathcal{P}(X, s)$.

Let $\text{VPRS}(\mathbf{V})$ denote the class of all varieties of pure recognizable \mathbf{V} -sets.

Of course, $(\text{VPRS}(\mathbf{V}), \subseteq)$ is a complete lattice. We shall now show that there is a natural correspondence between varieties of pure recognizable \mathbf{V} -sets and varieties of recognizable \mathbf{V} -sets.

DEFINITION 7.2 With any family $\mathcal{P} = \{\mathcal{P}(X, s)\}_{X, s}$ of pure recognizable \mathbf{V} -sets we associate the family of recognizable \mathbf{V} -sets $\mathcal{P}^r = \{\mathcal{P}^r(X)\}_X$ such that for each X ,

$$\mathcal{P}^r(X) = \{L \subseteq \mathbf{F}_{\mathbf{V}}(X) \mid (\forall s \in S) L_s \in \mathcal{P}(X, s)\}.$$

With any family $\mathcal{R} = \{\mathcal{R}(X)\}_X$ of recognizable \mathbf{V} -sets we associate the indexed family $\mathcal{R}^p = \{\mathcal{R}^p(X)\}_{X, s}$ of pure recognizable \mathbf{V} -sets such that for any X and $s \in S$,

$$\mathcal{R}^p(X, s) = \{L_s \mid L \in \mathcal{R}(X)\}.$$

Let us first note a few basic facts about these mappings. The notation $\langle T \rangle$ was introduced in Section 2.

LEMMA 7.3 Let $\mathcal{P} = \{\mathcal{P}(X, s)\}_{X, s}$ be a \mathbf{V} -VPRS and $\mathcal{R} = \{\mathcal{R}(X)\}_X$ be a \mathbf{V} -VRS. For any $X, s \in S$ and $T \subseteq \mathbf{F}_{\mathbf{V}}(X, s)$,

- (1) $T \in \mathcal{P}(X, s)$ iff $\langle T \rangle \in \mathcal{P}^r(X)$, and
- (2) $T \in \mathcal{R}^p(X, s)$ iff $\langle T \rangle \in \mathcal{R}(X)$.

Proof. If $T \in \mathcal{P}(X, s)$, then $\langle T \rangle \in \mathcal{P}^r(X)$ since $\langle T \rangle_s = T \in \mathcal{P}(X, s)$ and $\langle T \rangle_u = \emptyset \in \mathcal{P}(X, u)$ for every $u \in S, u \neq s$. On the other hand, if $\langle T \rangle \in \mathcal{P}^r(X)$, then $T = \langle T \rangle_s \in \mathcal{P}(X, s)$, and hence (1) holds.

To prove (2), assume first that $T \in \mathcal{R}^p(X, s)$. Then $T = L_s$ for some $L \in \mathcal{R}(X)$. If 1_s denotes the identity translation $F_{\mathbf{V}}(X, s) \rightarrow F_{\mathbf{V}}(X, s)$, then $\langle T \rangle = 1_s^{-1}(L) \in \mathcal{R}(X)$. On the other hand, if $\langle T \rangle \in \mathcal{R}(X)$, then $T = \langle T \rangle_s \in \mathcal{R}^p(X, s)$ by definition. \square

LEMMA 7.4 *The mappings $\mathcal{P} \mapsto \mathcal{P}^r$ and $\mathcal{R} \mapsto \mathcal{R}^p$ preserve inclusions. Moreover,*

- (1) *if $\mathcal{P} \in \text{VPRS}(\mathbf{V})$, then $\mathcal{P}^r \in \text{VRS}(\mathbf{V})$, and*
- (2) *if $\mathcal{R} \in \text{VRS}(\mathbf{V})$, then $\mathcal{R}^p \in \text{VPRS}(\mathbf{V})$.*

Proof. The first claim is completely obvious. Now, let $\mathcal{P} \in \text{VPRS}(\mathbf{V})$. That \mathcal{P}^r satisfies the conditions of Definition 5.2 follows easily from the assumption that \mathcal{P} satisfies the corresponding conditions of Definition 7.1. For example,

$$\begin{aligned} K, L \in \mathcal{P}^r(X) &\Rightarrow (\forall s \in S) K_s, L_s \in \mathcal{P}(X, s) \\ &\Rightarrow (\forall s \in S) K_s \cap L_s \in \mathcal{P}(X, s) \\ &\Rightarrow (\forall s \in S) (K \cap L)_s \in \mathcal{P}(X, s) \\ &\Rightarrow K \cap L \in \mathcal{P}^r(X), \end{aligned}$$

for any X and $K, L \subseteq F_{\mathbf{V}}(X)$. Similarly, if $L \in \mathcal{P}^r(X)$ and $\alpha \in \text{Tr}(\mathcal{F}_{\mathbf{V}}(X), s, s')$ for some X and $s, s' \in S$, then $L_{s'} \in \mathcal{P}(X, s')$ implies $\alpha^{-1}(L)_s = \alpha^{-1}(L_{s'}) \in \mathcal{P}(X, s)$, and hence $\alpha^{-1}(L) \in \mathcal{P}^r(X)$ as $\alpha^{-1}(L)_u = \emptyset \in \mathcal{P}(X, u)$ for every $u \in S \setminus \{s\}$.

Assertion (2) has a similar proof. \square

PROPOSITION 7.5 *The lattices $(\text{VPRS}(\mathbf{V}), \subseteq)$ and $(\text{VRS}(\mathbf{V}), \subseteq)$ are isomorphic since*

- (1) $\mathcal{P}^{rp} = \mathcal{P}$ for every $\mathcal{P} \in \text{VPRS}(\mathbf{V})$, and
- (2) $\mathcal{R}^{pr} = \mathcal{R}$ for every $\mathcal{R} \in \text{VRS}(\mathbf{V})$.

Proof. In view of Lemma 7.4 it suffices to prove (1) and (2), and these claims follow directly from Definition 7.2. For example, let $\mathcal{P} \in \text{VPRS}(\mathbf{V})$ and consider any X and $s \in S$. If $T \in \mathcal{P}(X, s)$, then $\langle T \rangle \in \mathcal{P}^r(X)$ by Lemma 7.3, and hence $T = \langle T \rangle_s$ belongs to \mathcal{P}^{rp} . Conversely: if $T \in \mathcal{P}^{rp}$, then there is an $L \in \mathcal{P}^r(X)$ such that $T = L_s$. But $L \in \mathcal{P}^r(X)$ means that $L_u \in \mathcal{P}(X, u)$ for every $u \in S$, and therefore, in particular, $T = L_s \in \mathcal{P}(X, s)$. Assertion (2) can be verified similarly. \square

Proposition 7.5 already implies that $(\text{VPRS}(\mathbf{V}), \subseteq)$ is isomorphic also to both the lattices $(\text{VFA}(\mathbf{V}), \subseteq)$ and $(\text{VFC}(\mathbf{V}), \subseteq)$ via $(\text{VRS}(\mathbf{V}), \subseteq)$, but we shall also exhibit direct isomorphisms. However, let us first consider generally syntactic congruences and algebras of subsets of a given sort.

Courcelle [6] defines the *syntactic congruence* of a subset $T \subseteq A_u$ of sort $u \in S$ of an Ω -algebra $\mathcal{A} = (A, \Omega)$ as the sorted equivalence $\sim^T = \langle \sim_s^T \rangle_{s \in S}$ on A such that for each $s \in S$ and any $a, b \in A$,

$$a \sim_s^T b \Leftrightarrow (\forall \alpha \in \text{Tr}(\mathcal{A}, s, u))(\alpha(a) \in T \leftrightarrow \alpha(b) \in T).$$

It is easy to see that \sim^T is the greatest congruence θ on \mathcal{A} such that θ_u saturates T . Let us call such congruences *pure syntactic congruences*.

The following lemma is quite obvious.

LEMMA 7.6 *Let $\mathcal{A} = (A, \Omega)$ be an Ω -algebra. Then $\sim^T = \approx^{(T)}$ for any subset $T \subseteq A_u$ of any given sort $u \in S$. On the other hand,*

$$\approx^L = \bigcap \{ \sim^{L_s} \mid s \in S \},$$

for any sorted subset $L = \langle L_s \rangle_{s \in S}$ of \mathcal{A} . □

Hence, any pure syntactic congruence is a syntactic congruence in our sense, while every syntactic congruence is the intersection of finitely many pure syntactic congruences.

The *syntactic algebra* of a subset $T \subseteq A_u$ of any sort $u \in S$ of an Ω -algebra $\mathcal{A} = (A, \Omega)$, is defined in [6] as the quotient algebra \mathcal{A}/\sim^T . Let us call an algebra *pure syntactic* if it is isomorphic to such a syntactic algebra.

PROPOSITION 7.7 *Any pure syntactic Ω -algebra is syntactic, and any syntactic Ω -algebra is a subdirect product of a finite family of pure syntactic Ω -algebras. Furthermore, every subdirectly irreducible Ω -algebra is pure syntactic.*

Proof. The first two assertions are immediate consequences of Lemma 7.6. If $\mathcal{A} = (A, \Omega)$ is subdirectly irreducible, then $\bigcap \{ \sim^{\{a\}} \mid a \in A_s, s \in S \} = \Delta_A$ implies that $\sim^{\{a\}} = \Delta_A$ for at least some $s \in S$ and $a \in A_s$, and hence $\mathcal{A} \cong \mathcal{A}/\sim^{\{a\}}$ is pure syntactic. □

COROLLARY 7.8 *Every \mathbf{V} -VFA is generated by pure syntactic algebras.* □

Let us return to pure recognizable \mathbf{V} -sets. The syntactic algebra $\mathcal{F}_{\mathbf{V}}(X)/\sim^T$ of a subset $T \subseteq F_{\mathbf{V}}(X, s)$ of sort some $s \in S$ of $\mathcal{F}_{\mathbf{V}}(X)$ is denoted simply $\text{PSA}(T)$. Note that $\text{PSA}(T) = \text{SA}(\langle T \rangle)$ by the first assertion of Lemma 7.6.

For any \mathbf{V} -VPRS \mathcal{P} , let \mathcal{P}^a be the \mathbf{V} -VFA generated by the pure syntactic algebras $\text{PSA}(T)$, where $T \in \mathcal{P}(X, s)$ for some X and $s \in S$.

LEMMA 7.9

- (1) $\mathcal{P}^a = \mathcal{P}^{ra}$ for any \mathbf{V} -VPRS \mathcal{P} , and
- (2) $\mathcal{R}^a = \mathcal{R}^{pa}$ for any \mathbf{V} -VRS \mathcal{R} .

Proof. To prove (1) it suffices to show that the syntactic algebras generating \mathcal{P}^a are in \mathcal{P}^{ra} , and conversely. For any X , $s \in S$ and $T \in \mathcal{P}(X, s)$, we have $\text{PSA}(T) = \text{SA}(\langle T \rangle) \in \mathcal{P}^{ra}$ since $\langle T \rangle \in \mathcal{P}^r$. Conversely, if $L \in \mathcal{P}^r(X)$ for some X , then $\text{SA}(L)$ is by Proposition 7.7 a subdirect product of the pure syntactic algebras $\text{PSA}(L_s)$ ($s \in S$). Because $L_s \in \mathcal{P}(X, s)$, and therefore $\text{PSA}(L_s) \in \mathcal{P}^a$, for every $s \in S$, this means that $\text{SA}(L) \in \mathcal{P}^a$. Assertion (2) has an equally straightforward proof. \square

Now it is clear that $\mathcal{P} \mapsto \mathcal{P}^a$ defines an isomorphism from $(\text{VPRS}(\mathbf{V}), \subseteq)$ to $(\text{VFA}(\mathbf{V}), \subseteq)$. In fact, it is the composition of the two isomorphisms $\mathcal{P} \mapsto \mathcal{P}^r$ and $\mathcal{R} \mapsto \mathcal{R}^a$. Its converse can be defined explicitly as follows: for any \mathbf{V} -VFA \mathbf{K} , let \mathbf{K}^p be the family of pure recognizable \mathbf{V} -sets such that for any X , $s \in S$ and $T \subseteq \text{F}_{\mathbf{V}}(X, s)$, $T \in \mathbf{K}^p(X, s)$ iff $\text{PSA}(T) \in \mathbf{K}$.

Corresponding to Lemma 7.9 the following facts hold.

LEMMA 7.10 For any \mathbf{V} -VFA \mathbf{K} , (1) $\mathbf{K}^p = \mathbf{K}^{rp}$, and (2) $\mathbf{K}^r = \mathbf{K}^{pr}$.

Proof. To prove (1) we note that for any X , $s \in S$ and $T \subseteq \text{F}_{\mathbf{V}}(X, s)$,

$$T \in \mathbf{K}^p(X, s) \Leftrightarrow \text{PSA}(T) \in \mathbf{K} \Leftrightarrow \text{SA}(\langle T \rangle) \in \mathbf{K} \Leftrightarrow \langle T \rangle \in \mathbf{K}^r \Leftrightarrow T \in \mathbf{K}^{rp}.$$

Now (2) follows since $\mathbf{K}^{pr} = \mathbf{K}^{rpr} = \mathbf{K}^r$ by Proposition 7.5. \square

Next we consider pure recognizable \mathbf{V} -sets and \mathbf{V} -congruences. Proposition 7.5 and the Variety Theorem yield the isomorphisms $\text{VPRS}(\mathbf{V}) \rightarrow \text{VFC}(\mathbf{V}), \mathcal{P} \mapsto \mathcal{P}^{rc}$ and $\text{VFC}(\mathbf{V}) \rightarrow \text{VPRS}(\mathbf{V}), \Gamma \mapsto \Gamma^{cp}$, via $\text{VRS}(\mathbf{V})$, but we can also define them directly as follows.

For any \mathbf{V} -VPRS \mathcal{P} , let \mathcal{P}^c be the family of \mathbf{V} -congruences such that for each X , $\mathcal{P}^c(X)$ is the filter of $\text{FCon}(\mathcal{F}_{\mathbf{V}}(X))$ generated by the pure syntactic congruences \sim^T , where $T \in \mathcal{P}(X, s)$ for some $s \in S$. Conversely, for any \mathbf{V} -VFC Γ , let Γ^p be the family of pure recognizable \mathbf{V} -sets such that for any X and $s \in S$,

$$\Gamma^p(X, s) = \{T \subseteq \text{F}_{\mathbf{V}}(X, s) \mid \sim^T \in \Gamma(X)\}.$$

LEMMA 7.11

(1) $\mathcal{P}^c = \mathcal{P}^{rc}$ for any \mathbf{V} -VPRS \mathcal{P} , and

(2) $\mathcal{R}^c = \mathcal{R}^{pc}$ for any \mathbf{V} -VRS \mathcal{R} .

Proof. To prove (1), we show that for any X , the generators of $\mathcal{P}^c(X)$ are in $\mathcal{P}^{rc}(X)$, and the generators of $\mathcal{P}^{rc}(X)$ are in $\mathcal{P}^c(X)$.

For any $s \in S$ and $T \in \mathcal{P}(X, s)$, $\langle T \rangle \in \mathcal{P}^r(X)$ and $\sim^T = \approx^{\langle T \rangle} \in \mathcal{P}^{rc}$. On the other hand, if $L \in \mathcal{P}^r(X)$, then $L_s \in \mathcal{P}(X, s)$ for each $s \in S$, and hence $\approx^{L_s} \in \mathcal{P}^c(X)$ by Lemma 7.6. Assertion (2) can be verified similarly. \square

LEMMA 7.12 For any \mathbf{V} -VFC Γ , (1) $\Gamma^p = \Gamma^{rp}$, and (2) $\Gamma^r = \Gamma^{pr}$.

Proof. To prove (1), it suffices to note that for any X , $s \in S$ and $T \subseteq F_{\mathbf{V}}(X, s)$, $T \in \Gamma^p(X, s) \Leftrightarrow \sim^T \in \Gamma(X) \Leftrightarrow \approx^{(T)} \in \Gamma(X) \Leftrightarrow \langle T \rangle \in \Gamma^r(X) \Leftrightarrow T \in \Gamma^{rp}(X)$, where Lemma 7.3 is used in the last step.

Assertion (2) follows from (1) and Proposition 7.5: $\Gamma^{pr} = \Gamma^{rpr} = \Gamma^r$. \square

Acknowledgements. We want to thank the referee for some useful remarks.

References

- [1] J. ALMEIDA, *On pseudovarieties, varieties of languages, pseudoidentities and related topics*, Algebra Universalis, **27** (1990), 333–350.
- [2] G. BIRKHOFF and J.D. LIPSON, *Heterogeneous algebras*, J. Comb. Theory, **8** (1970), 115–133.
- [3] S. BURRIS and H.P. SANKAPPANAVAR, *A Course in Universal Algebra*, Springer-Verlag, New York, 1981.
- [4] P.M. COHN, *Universal Algebra* (2nd ed.), Mathematics and its Applications 6, D. Reidel Publishing Co., Dordrecht-Boston, Mass., 1981.
- [5] B. COURCELLE, *On recognizable sets and tree automata*, in: M. Nivat, H. Aït-Kaci (eds.), Resolution of Equations in Algebraic Structures. Vol. 1, Academic Press, New York, 1989, 93–126.
- [6] B. COURCELLE, *Basic notions of universal algebra for language theory and graph grammars*, Theoret. Comput. Sci., **163** (1996), 1–54.
- [7] K. DENECKE and SH.L. WISMATH, *Universal Algebra and Applications in Theoretical Computer Science*, Chapman & Hall/CRC, Boca Raton, 2002.
- [8] S. EILENBERG, *Automata, Languages, and Machines*, Vol. B., Academic Press, New York, 1976.
- [9] J. ENGELFRIET and E.M. SCHMIDT, *IO and OI, Part I*, J. Comput. Systems Sci., **15** (1977), 328–353; Part II, *ibidem* **16** (1978), 67–99.
- [10] Z. ÉSIK, *A variety theory of trees and theories*, Publ. Math. Debrecen, **54** (Supplement) (1999), 711–762.
- [11] Z. ÉSIK and P. WEIL, *On logically defined recognizable tree languages*, in: P.K. Pandya, J. Radhakrishnan (eds.) Foundations of Software Technology and Theoretical Computer Science, (Proc. Conf. FSTTCS'03, Bombay 2003), Lect. Notes in Comput. Sci 2914, Springer-Verlag, Heidelberg 2003, 195–207.
- [12] P.J. HIGGINS, *Algebras with a scheme of operators*, Math. Nachr., **27** (1963), 115–132.
- [13] H. LUGOWSKI, *Grundzüge der Universellen Algebra*, B.G. Teubner Verlagsgesellschaft, Leipzig, 1976.
- [14] T.S.E. MAIBAUM, *A generalized approach to formal languages*, J. Comput. Systems Sci., **8** (1974), 409–439.

-
- [15] K. MEINKE and J.V. TUCKER, *Universal algebra*, in: S. Abramsky, D. Gabbay, T.S. Maibaum (eds.), *Handbook of Logic in Computer Science*, Vol. 1, Clarendon Press, Oxford, 1992, 189–411.
 - [16] J. MEZEI and J.B. WRIGHT, *Algebraic automata and context-free sets*, *Information and Control*, **11** (1967), 3–29.
 - [17] J.E. PIN, *Varieties of formal languages*, North Oxford Academic Publishers, Oxford, 1986.
 - [18] J.E. PIN, *Syntactic semigroups*, in: G. Rozenberg, A. Salomaa (eds.), *Handbook of Formal Languages*, Vol. 1, Springer-Verlag, Berlin, 1997, 679–746.
 - [19] S. SALEHI, *Varieties of Tree Languages*, TUCS Dissertaions 64, Turku Centre for Computer Science, Turku 2005.
 - [20] S. SALEHI and M. STEINBY, *Tree Algebras and Regular Tree Languages*, *Theoret. Comput. Sci.*, **377** (2007), 1–24.
 - [21] M. STEINBY, *Syntactic algebras and varieties of recognizable sets*, in: M.C. Gaudel, J.P. Jouannaud (eds.), *Les Arbres en Algèbre et en Programmation* (Proc. 4th CAAP, Lille 1979), University of Lille, Lille 1979, 226–240.
 - [22] M. STEINBY, *A theory of tree language varieties*, in: M. Nivat, A. Podelski (eds.), *Tree Automata and Languages*, North-Holland, Amsterdam, 1992, 57–81.
 - [23] M. STEINBY, *General varieties of tree languages*, *Theoret. Comput. Sci.*, **205** (1998), 1–43.
 - [24] M. STEINBY, *Algebraic characterizations of regular tree languages*, in: V.B. Kudryavtsev, I.G. Rosenberg (eds.) *Structural Theory of Automata, Semigroups, and Universal Algebra*, NATO Science Series, II. Mathematics, Physics and Chemistry, Vol. 207, Springer, Dordrecht 2005, 381–432.
 - [25] D. THÉRIEN, *Recognizable languages and congruences*, *Semigroup Forum*, **23** (1981), 371–373.
 - [26] T. WILKE, *An algebraic characterization of frontier testable tree languages*, *Theoret. Comput. Sci.*, **154** (1996), 85–106.