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On Gödel's 'Much Weaker' Assumption

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ABSTRACT

Gödelian sentences of a sufficiently strong and recursively enumerable theory, constructed in Gödel's ground-breaking paper of 1931 on the incompleteness theorems, are unprovable if the theory is consistent; however, they could be refutable. These sentences are independent when the theory is so-called ω-consistent; a notion introduced by Gödel, which is stronger than (simple) consistency, but 'much weaker' than soundness. Gödel goes to great lengths to show in detail that ω-consistency is stronger than consistency, but never shows, or seems to forget to say, why it is much weaker than soundness. In this paper, we study this proof-theoretic notion and compare some of its properties with those of consistency and (variants of) soundness.

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1. Introduction

In the penultimate paragraph of the first section of his celebrated paper on the incompleteness theorems, Gödel (*1931*, p. 151) wrote:

> The method of proof just explained can clearly be applied to any formal system that, first, [...] and in which, second, every provable formula is true in the interpretation considered. The purpose of carrying out the above proof with full precision in what follows is, among other things, to replace the second of the assumptions just mentioned by a purely formal and much weaker one.

He began the next section with the sentence, 'We now proceed to carry out with full precision the proof sketched above'. It is clear then that Gödel (*1931*) sketched his proof of the first incompleteness theorem in Section 1 for the system of *Principia Mathematica*, and then noted that his method of proof works for any formal system that, first, is *sufficiently strong* (in today's terminology) and, second, is *sound* (with respect to the standard model of natural numbers, see, e.g. §2 of *Isaacson 2011* for the terminology). He then said that in the rest of the article the proof would be carried out with full precision, while the second assumption (that of soundness) was replaced by a 'purely formal and much weaker one'. This assumption was called ω-consistency by him (*Gödel 1931*, p. 173); see Definition 2.1 below. A question pursued in this paper is the following:

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Dedicated to the loving memory of Magnus Steinby (1941–2021). †

Why is the purely formal notion of ω-consistency much weaker than soundness?[1]

One possible answer could be the pure formality of ω-consistency itself! Gödel knew that soundness (or truth) is not purely formal (what we know today from Tarski's Undefinability Theorem); see, e.g. *Murawski 1998*. And, ω-consistency is purely formal (arithmetically definable, see Definitions 2.7 and 2.8 below). Since soundness implies ω-consistency, and the latter is definable while the former is not, then ω-consistency should be (much) weaker than soundness. Could Gödel (*1931*) have meant in the penultimate paragraph 'to replace the second of the assumptions just mentioned by a purely formal and (thus) much weaker one?' In other words, could his reason for the weakness of ω-consistency in comparison with soundness be the pure formality (arithmetical definability) of the former (and *un*definability of the latter)?

On the other hand, the independence of the Gödelian sentences can be guaranteed by much weaker assumptions (much weaker than ω-consistncy!). Indeed, 1-consistency is more than enough; see *Isaacson 2011*, §§5.1 for the definition of 1-consistency, its sufficiency, and its being weaker than ω-consistency, also *Isaacson 2011*, §6, for a sufficient condition weaker than 1-consistency. Even Gödel (*1931*) mentioned in the very last page that 'we can, in [the first incompleteness theorem], replace the assumption of ω-consistency by the following: The proposition "κ is inconsistent" is not κ-PROVABLE'; this condition, which is equivalent to the consistency of the theory with its (standard) Consistency Statement, is also necessary for the independence of the Gödelian sentences, see *Isaacson 2011*, Theorem 35. These stronger (than simple consistency) assumptions were all removed by Rosser (*1936*) who showed the independence of some other sentences from consistent theories (that are recursively enumerable and contain sufficient arithmetic).

Before going to technical details, let us quote Smoryński (*1985*, p. 158, Remark) about ω-consistency: 'One weakness of Gödel's original work was his introduction of the semantic notion of ω-consistency. I find this notion to be pointless, but I admit many proof theorists take it seriously.' It is notable that some prominent logicians, of the caliber of Henkin (*1954*) studied, and even generalized, the concept of ω-consistency, which is a *syntactic* (purely formal, proof-theoretic) notion; not 'semantic'! (see Remark 3.3 below). However, we can agree with Smoryński (*1985*) that ω-consistency could be 'pointless', and may be dismissed.

2. ω-Consistency and Some of Its Properties

Let us fix a sufficiently strong theory $\mathbb{P}$ over an arithmetical language (which contains $+$, $\times$, and possibly some other constant, relation, or function symbols). This could be Peano's Arithmetic, which is more than enough, or some of its weaker fragments, such as $\mathrm{I}\Sigma_1$.

All our theories are (usually RE) sets of arithmetical sentences that contain $\mathbb{P}$.

Let us be given a fixed Gödel coding and arithmetization by which we have the provability predicate $\mathtt{Pr}_T(x)$, for a fixed coding of the theory T, saying that '(the sentence with code) x is T-provable'. We assume familiarity with the notions of Σ_m and Π_m formulas.

[1] See also *Isaacson 2011*, p. 141, the paragraph after the proof of Proposition 19.

Definition 2.1 (ω-Consistency): The theory T is called ω-consistent, when there is *no* formula $\varphi(x)$, with exactly one free variable x, such that the negation of the universal closure of φ and all the numerical instances of φ are T-provable (i.e. $T \vdash \neg\forall x\varphi(x)$ and $T \vdash \varphi(\overline{n})$ for each $n \in \mathbb{N}$; where $\overline{n}$ denotes the standard term that represents n).

Example 2.2 (of an ω-consistent and an ω-inconsistent theory): Every sound theory is ω-consistent; see *Isaacson 2011*, Theorem 16. To see a natural ω-*in*consistent theory, let us consider the negation of the (formal) Induction Principle. For a formula $\varphi(x)$, the formal Induction Principle of φ is

$$\mathrm{IND}_\varphi : \quad \varphi(\overline{0}) \wedge \forall x\,[\varphi(x) \to \varphi(x+\overline{1})] \longrightarrow \forall x\,\varphi(x).$$

It is known that the IND of formulas with smaller complexity do not necessarily imply the IND of formulas with higher complexity. So, $\neg\mathrm{IND}_\varphi$ could be consistent with some weak arithmetical theories, for some sufficiently complex formula φ. We show that $\neg\mathrm{IND}_\varphi$ entails an ω-inconsistency. First, note that $\neg\mathrm{IND}_\varphi \vdash \neg\forall x\varphi(x)$. Second, we have $\neg\mathrm{IND}_\varphi \vdash \varphi(\overline{n}) \to \varphi(\overline{n+1})$ for every $n \in \mathbb{N}$, and so by (meta-)induction on n one can show $\neg\mathrm{IND}_\varphi \vdash \varphi(\overline{n})$ for every $n \in \mathbb{N}$, noting that $\neg\mathrm{IND}_\varphi \vdash \varphi(\overline{0})$. Therefore, $\neg\mathrm{IND}_\varphi$ is ω-inconsistent.

We will use the following result of *Isaacson 2011*, Theorem 21, which is the ω-version of Lindenbaum's lemma:

Proposition 2.3 (*Isaacson 2011*: $\omega\mathsf{-Con}_T \Longrightarrow \forall\psi\colon \omega\mathsf{-Con}_{T+\psi} \vee \omega\mathsf{-Con}_{T+\neg\psi}$): *If T is ω-consistent, then for every sentence ψ either $T+\psi$ or $T+\neg\psi$ is ω-consistent.*

Proof: Assume (for the sake of a contradiction) that both theories $T+\psi$ and $T+\neg\psi$ are ω-inconsistent. Then for some formulas $\alpha(x)$ and $\beta(x)$ we have $T+\psi \vdash \neg\forall x\,\alpha(x)$ and $T+\psi \vdash \alpha(\overline{n})$ for each $n \in \mathbb{N}$, also $T+\neg\psi \vdash \neg\forall x\,\beta(x)$ and $T+\neg\psi \vdash \beta(\overline{n})$ for each $n \in \mathbb{N}$. By Deduction Theorem we have $T \vdash \neg\forall x\,[\psi \to \alpha(x)]$ and $T \vdash \neg\forall x\,[\neg\psi \to \beta(x)]$, and so by Classical Logic we have (I) $T \vdash \neg\forall x\,([\psi \to \alpha(x)] \wedge [\neg\psi \to \beta(x)])$. Again, by Deduction Theorem, for every $n \in \mathbb{N}$ we have (II) $T \vdash [\psi \to \alpha(\overline{n})] \wedge [\neg\psi \to \beta(\overline{n})]$. Now, (I) and (II) imply that T is not ω-consistent, which contradicts the assumption. ∎

Corollary 2.4 (Consistency of ω-consistent theories with PA): *Every ω-consistent theory is consistent with Peano's Arithmetic.*

Proof: Suppose that T is an ω-consistent theory. By Proposition 2.3 and Example 2.2, $T+\mathrm{IND}_\varphi$, for an arbitrary formula $\varphi(x)$, is ω-consistent too. So is the theory $T+\{\mathrm{IND}_{\varphi_1}, \ldots, \mathrm{IND}_{\varphi_n}\}$, for any finite set of formulas $\{\varphi_1(x), \ldots, \varphi_n(x)\}$. Thus, T (which contains $\mathbb{P}$) is consistent with Peano's Arithmetic. ∎

We next observe that ω-consistent theories remain ω-consistent when extended by any true Σ_3-sentence. This was first proved for true Π_1-sentences in *Isaacson 2011*, Theorem 22, with a proof attributed to '(*letter from Georg Kreisel 4 April 2005*)'. Let us recall that our base theory $\mathbb{P}$ is Σ_1-complete, i.e. can prove every true Σ_1-sentence.

Theorem 2.5 (ω–$\mathtt{Con}_T \wedge \sigma \in \Sigma_3$–$\mathrm{Th}(\mathbb{N}) \Longrightarrow \omega$–$\mathtt{Con}_{T+\sigma} \wedge \neg\omega$–$\mathtt{Con}_{T+\neg\sigma}$): *If T is an ω-consistent theory and σ is a true Σ_3-sentence, then $T+\sigma$ is ω-consistent and $T+\neg\sigma$ is ω-inconsistent.*

Proof: Let $\sigma = \exists x\, \pi(x)$ for a Π_2-formula π. Since σ is true, then there exists some $k \in \mathbb{N}$ such that $\mathbb{N} \vDash \pi(\overline{k})$. Let $\pi(\overline{k}) = \forall y\, \theta(y)$ for some Σ_1-formula θ. Then for every $n \in \mathbb{N}$ we have $\mathbb{N} \vDash \theta(\overline{n})$. So, by the Σ_1-completeness of $\mathbb{P}$ we have $\mathbb{P} \vdash \theta(\overline{n})$ for each $n \in \mathbb{N}$. Now, from $\neg\sigma \vdash \neg\pi(\overline{k})$ we have $\neg\sigma \vdash \neg\forall x\, \theta(x)$. Thus, $\mathbb{P}+\neg\sigma$ is ω-inconsistent; so is $T+\neg\sigma$. Since T is ω-consistent, then by Proposition 2.3, $T+\sigma$ must be ω-consistent. ■

Later, we will see that this result is optimal: adding a true Π_3-sentence to an ω-consistent theory does not necessarily result in an ω-consistent theory (see Corollary 2.13 below). Let us now note that ω-consistency implies Π_3-soundness.

Corollary 2.6 (ω–$\mathtt{Con}_T \Longrightarrow \Pi_3$–$\mathtt{Sound}_T$): *Every ω-consistent theory is Π_3-sound, i.e. every provable Π_3-sentence of it is true.*

Proof: If T is ω-consistent, and π is a T-provable Π_3-sentence, then π must be true, since otherwise $\neg\pi$ would be a true Σ_3-sentence, and so $T+\neg\pi$ would be ω-consistent by Theorem 2.5, but this is a contradiction since $T+\neg\pi$ is inconsistent by T-provability of π. ■

We now note that the notion of ω-consistency is arithmetically definable.

Definition 2.7 ($\mho_T(\ulcorner\varphi\urcorner)$: the formula φ is a witness for the ω-inconsistency of T): Let $\mathtt{Frm}_1(x)$ say that 'x is (the Gödel code of) a formula with exactly one free variable'. Let $\mho_T(x)$ say that 'x is (the code of) a formula with exactly one free variable, and the negation of the universal closure of (the formula coded by) x and also every (numerical) instance of (the formula coded by) x is T-provable'. I.e. $\mho_T(\ulcorner\varphi\urcorner)$ is the following formula: $\mathtt{Frm}_1(\ulcorner\varphi\urcorner) \wedge \mathtt{Pr}_T(\ulcorner\neg\forall v\varphi(v)\urcorner) \wedge \forall w\, \mathtt{Pr}_T(\ulcorner\varphi(\dot{w})\urcorner)$. Here $\ulcorner\alpha\urcorner$ (which is a term in the language of $\mathbb{P}$) denotes the Gödel code of the expression α, and $\dot{w}$ is Feferman's dot notation.

Definition 2.8 (ω–$\mathtt{Con}_T$): Let ω–$\mathtt{Con}_T$ be the sentence $\neg\exists x\, \mho_T(x)$, where $\mho_T(x)$ is defined in Definition 2.7.

We note that when T is an RE theory, then $\mathtt{Pr}_T(x)$ is a Σ_1-formula, so $\mho_T(x)$ is a Π_2-formula, thus ω–$\mathtt{Con}_T$ is a Π_3-sentence. As far as I know, the first proof of the weakness of ω-consistency with respect to soundness appeared in print in *Kreisel 1955* and is referred to as '(*Kreisel 1955*)' in *Isaacson 2011*, Proposition 19.

Definition 2.9 (Kreiselian Σ_3-Sentences of T, κ: I am ω-inconsistent with T): For a theory T, any Σ_3-sentence κ that satisfies $\mathbb{P} \vdash \kappa \leftrightarrow \neg\omega$–$\mathtt{Con}_{T+\kappa}$ is called a Kreiselian sentence of T.

If one thought, mistakenly, that ω-consistency is equivalent to soundness, then Kreiselian sentences correspond to the Liar sentences. Now, Kreisel's proof of the non-equivalence of ω-consistency with soundness corresponds to the classical proof of Tarski's Undefinability Theorem. Let us note that by Diagonal Lemma there exist some Kreiselian sentences for any RE theory, be it ω-consistent or not; see *Isaacson 2011*, Proposition 19.

Theorem 2.10 ($\omega\text{−}\mathsf{Con}_T \Longrightarrow \omega\text{−}\mathsf{Con}_{T+\kappa}$ & $\mathbb{N}\nvDash\kappa$): *If T is RE and ω-consistent, and κ is a Kreiselian sentence of T, then κ is false and $T+\kappa$ is ω-consistent.*

Proof: If κ were true, then by Definition 2.9 and the soundness of $\mathbb{P}$, the theory $T+\kappa$ would be ω-inconsistent. But by Theorem 2.5, and the assumed truth of the Σ_3-sentence κ, the theory $T+\kappa$ should be ω-consistent; a contradiction. Thus, κ is false; and so, by the soundness of $\mathbb{P}$, the theory $T+\kappa$ is ω-consistent. ■

So, for an RE ω-consistent theory T and a Kreiselian sentence κ of T, the theory $T+\kappa$ is ω-consistent but not sound (since κ is false); $T+\kappa$ is not even Σ_3-sound.

Corollary 2.11 ($\omega\text{−}\mathsf{Con}_T \not\Longrightarrow \Sigma_3\text{−}\mathsf{Sound}_T$): *$\omega$-consistency does not imply Σ_3-soundness.*

Theorem 2.12 ($\Sigma_m\text{−}\mathsf{Sound}_T \not\Longrightarrow \omega\text{−}\mathsf{Con}_T$): *For any $m\in\mathbb{N}$, Σ_m-soundness does not imply ω-consistency.*

Proof: It is rather easy to see that Σ_m-soundness (the truth of provable Σ_m-sentences) is equivalent to consistency with $\Pi_m\text{−Th}(\mathbb{N})$, the set of true Π_m-sentences. By *Salehi and Seraji 2017*, Theorem 2.5, there exists a true Π_{m+1}-sentence γ such that $\mathbb{P}+\Pi_m\text{−Th}(\mathbb{N})\nvdash\gamma$. So, the theory $U=\mathbb{P}+\Pi_m\text{−Th}(\mathbb{N})+\neg\gamma$ is consistent. We show that U is not ω-consistent. Let $\gamma=\forall x\,\sigma(x)$ for some Σ_m-formula σ. By the truth of γ we have $\Pi_m\text{−Th}(\mathbb{N})\vdash\sigma(\overline{n})$ for each $n\in\mathbb{N}$. So, we have $U\vdash\neg\forall x\,\sigma(x)$ and $U\vdash\sigma(\overline{n})$ for each $n\in\mathbb{N}$. Thus, U is not ω-consistent, but it is Σ_m-sound (being consistent with $\Pi_m\text{−Th}(\mathbb{N})$). However, U is not RE; let us consider its sub-theory $T=\mathbb{P}+\neg\forall x\,\sigma(x)+\{\sigma(\overline{n})\}_{n\in\mathbb{N}}$. The theory T is RE and Σ_m-sound, but not ω-consistent. ■

Therefore, while soundness implies ω-consistency, Σ_m-soundness, even for large m's, does not imply ω-consistency. We can now show the optimality of Theorem 2.5.

Corollary 2.13 ($\omega\text{−}\mathsf{Con}_T \wedge \pi\in\Pi_3\text{−Th}(\mathbb{N}) \not\Longrightarrow \omega\text{−}\mathsf{Con}_{T+\pi}$): *Adding a true Π_3-sentence to an ω-consistent theory does not necessarily result in an ω-consistent theory.*

Proof: Let κ_0 be a Kreiselian sentence of $\mathbb{P}$. Then, by Theorem 2.10, the theory $T_0=\mathbb{P}+\kappa_0$ is ω-consistent and $\neg\kappa_0$ is a true Π_3-sentence. But $T_0+\neg\kappa_0$ is not even consistent. ■

Finally, we can show that adding a Kreiselian sentence or its negation to a sound theory results, in both cases, in an ω-consistent theory (cf. Theorem 3.7 below).

Corollary 2.14 ($\mathbb{N}\vDash T \Longrightarrow \omega\text{−}\mathsf{Con}_{T+\kappa} \wedge \omega\text{−}\mathsf{Con}_{T+\neg\kappa}$): *If κ is a Kreiselian sentence of a sound RE theory T, then both $T+\kappa$ and $T+\neg\kappa$ are ω-consistent.*

Proof: The theory $T+\neg\kappa$ is sound, and $T+\kappa$ is ω-consistent by Theorem 2.10. ■

3. Some Syntactic Properties of ω-Consistency

Let us begin this section, like the previous one, with another interesting result of Isaacson (*2011*, Theorem 20); see *Salehi and Seraji 2017*, Proposition 3.2, for a generalization.

Proposition 3.1 (*Isaacson 2011*: $\boldsymbol{\omega}$–$\mathtt{Con}_T \wedge \mathtt{Complete}_T \Longrightarrow T = \mathrm{Th}(\mathbb{N})$): *True Arithmetic,* $\mathrm{Th}(\mathbb{N})$*, is the only ω-consistent theory which is complete.*

Proof: Let T be a complete and ω-consistent theory; T is Π_3-sound by Corollary 2.6. So, we have $(\mathcal{C}_2)$ $T \vdash \Sigma_2\text{–}\mathrm{Th}(\mathbb{N}) + \Pi_2\text{–}\mathrm{Th}(\mathbb{N})$; since if $\eta \in \Sigma_2\text{–}\mathrm{Th}(\mathbb{N}) + \Pi_2\text{–}\mathrm{Th}(\mathbb{N})$ and $T \nvdash \eta$, then $T \vdash \neg\eta$, by the completeness of T, which would contradict the Π_3-soundness of T. We now show, by induction on $m \geqslant 2$, that $(\mathcal{C}_m)$ $T \vdash \Sigma_m\text{–}\mathrm{Th}(\mathbb{N}) + \Pi_m\text{–}\mathrm{Th}(\mathbb{N})$. For proving $(\mathcal{C}_m \Rightarrow \mathcal{C}_{m+1})$ suppose that $(\mathcal{C}_m)$ holds. It is easy to see that $\Pi_m\text{–}\mathrm{Th}(\mathbb{N}) \vdash \Sigma_{m+1}\text{–}\mathrm{Th}(\mathbb{N})$; so by $(\mathcal{C}_m)$ we already have $T \vdash \Sigma_{m+1}\text{–}\mathrm{Th}(\mathbb{N})$. We now show $T \vdash \Pi_{m+1}\text{–}\mathrm{Th}(\mathbb{N})$. Let π be a true Π_{m+1}-sentence; let $\pi = \forall x\, \sigma(x)$ for some Σ_m-formula σ. For every $n \in \mathbb{N}$ we have $\mathbb{N} \vDash \sigma(\overline{n})$, so $\Sigma_m\text{–}\mathrm{Th}(\mathbb{N}) \vdash \sigma(\overline{n})$, thus by $(\mathcal{C}_m)$ we have $T \vdash \sigma(\overline{n})$. Now, the ω-consistency of T implies $T \nvdash \neg\forall x\, \sigma(x)$; so $T \vdash \forall x\, \sigma(x)$ from the completeness of T, thus $T \vdash \pi$. ■

Corollary 3.2 ($\lim_{\subseteq}$ $\boldsymbol{\omega}$–$\mathtt{Con} \neq \boldsymbol{\omega}$–$\mathtt{Con}$): *The limit (union) of a chain of ω-consistent theories is not necessarily ω-consistent.*

Proof: Let κ_0 be a Kreiselian sentence of $\mathbb{P}$ and put $T_0 = \mathbb{P} + \kappa_0$. Then T_0 is ω-consistent by Theorem 2.10. Now, by Proposition 2.3 one can expand T_0 in stages $T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots$ in a way that each T_m is ω-consistent and their union $T^* = \bigcup_m T_m$ is complete. But by Proposition 3.1, T^* cannot be ω-consistent since $\mathbb{N} \nvDash \kappa_0$ by Theorem 2.10 and so $T^* \neq \mathrm{Th}(\mathbb{N})$. Thus, the limit T^* of the chain $\{T_m\}_m$ of ω-consistent theories is not ω-consistent. ■

Remark 3.3 (Non-Semanticity of $\boldsymbol{\omega}$-Consistency): Proposition 3.1 enables us to show that ω-consistency is not a semantic (model-theoretic) notion. Assume, for the sake of a contradiction, that for a class $\mathscr{C}$ of structures (over the language of $\mathbb{P}$) and for every theory T, (♠) T is ω-consistent if and only if $\mathcal{M} \vDash T$ for some $\mathcal{M} \in \mathscr{C}$.

A candidate for such a $\mathscr{C}$ that comes to mind naturally is the class of ω-type structures: a model $\mathfrak{A}$ is called ω-type, when there is no formula $\varphi(x)$ such that $\mathfrak{A} \vDash \neg\forall x\, \varphi(x)$ and at the same time $\mathfrak{A} \vDash \varphi(\overline{n})$ for every $n \in \mathbb{N}$. It is clear that if a theory has an ω-type model, then it is an ω-consistent theory.

For showing the impossibility of (♠), take T_0 to be an unsound ω-consistent theory (as in e.g. the proof of Corollary 3.2) and assume that for $\mathcal{M}_0 \in \mathscr{C}$ we have $\mathcal{M}_0 \vDash T_0$. Then, the full first-order theory $\mathrm{Th}(\mathcal{M}_0)$ of $\mathcal{M}_0$, the set of all sentences that are true in $\mathcal{M}_0$, is a complete ω-consistent theory. So, by Proposition 3.1, $\mathrm{Th}(\mathcal{M}_0)$ should be equal to $\mathrm{Th}(\mathbb{N})$; thus $\mathcal{M}_0 \equiv \mathbb{N}$ whence $\mathbb{N} \vDash T_0$, a contradiction.[2]

We now show that Theorem 2.5 can be formalized in $\mathbb{P}$.

[2] It can be shown that a structure is ω-type if and only if it is an elementary extension of $\mathbb{N}$; Proposition 3.1 can also be proved by the Tarski-Vaught test for elementarity: If $\mathfrak{M}$ is a model of a complete ω-consistent theory, then for any formula $\varphi(x)$, $\mathfrak{M} \vDash \exists x \varphi(x)$ implies the existence of some standard $n \in \mathbb{N}$ with $\mathfrak{M} \vDash \varphi(\overline{n})$.

Theorem 3.4 ($\sigma \in \Sigma_3 \Longrightarrow \mathbb{P} \vdash \sigma \wedge \omega\text{--}\mathtt{Con}_T \rightarrow \omega\text{--}\mathtt{Con}_{T+\sigma}$): *For every Σ_3-sentence σ and any theory T we have $\mathbb{P} \vdash \sigma \wedge \omega\text{--}\mathtt{Con}_T \rightarrow \omega\text{--}\mathtt{Con}_{T+\sigma}$.*

Proof: The proof of Theorem 2.5 (which was based on Proposition 2.3) can be formalized in $\mathbb{P}$ with some hard work. We now present a more direct proof for Theorem 2.5 (without appealing to Proposition 2.3) whose formalizability in $\mathbb{P}$ is straightforward. Suppose that $\omega\text{--}\mathtt{Con}_T$ and that σ is a true Σ_3-sentence. If $\neg\omega\text{--}\mathtt{Con}_{T+\sigma}$ then for some formula $\varphi(x)$ we have $T+\sigma \vdash \neg\forall x\, \varphi(x)$ and $T+\sigma \vdash \varphi(\overline{n})$ for every $n \in \mathbb{N}$. Let $\sigma = \exists x\, \pi(x)$ for a Π_2-formula π. Since σ is true, then there exists some $u (\in \mathbb{N})$ such that $\pi(u)$ is true. Let $\pi(u) = \forall y\, \theta(y)$ for some Σ_1-formula θ. Then $\theta(z)$ is true for every z. So, by the Σ_1-completeness of T we have (♣) $T \vdash \theta(\overline{n})$ for each $n \in \mathbb{N}$. For reaching to a contradiction, we show that T is ω-inconsistent and the formula $\theta(x) \wedge [\pi(u) \rightarrow \varphi(x)]$ is a witness for that. By Deduction Theorem we have $T \vdash \sigma \rightarrow \neg\forall x\, \varphi(x)$ and so $T \vdash \pi(u) \rightarrow \neg\forall x\, [\pi(u) \rightarrow \varphi(x)]$ therefore $T \vdash \neg\forall y\, \theta(y) \vee \neg\forall x\, [\pi(u) \rightarrow \varphi(x)]$, thus (i) $T \vdash \neg\forall x\, (\theta(x) \wedge [\pi(u) \rightarrow \varphi(x)])$. On the other hand, for every $n \in \mathbb{N}$ we have $T \vdash \pi(u) \rightarrow \varphi(\overline{n})$, which by (♣) implies that (ii) $T \vdash \theta(\overline{n}) \wedge [\pi(u) \rightarrow \varphi(\overline{n})]$ for each $n \in \mathbb{N}$. Thus, by (i) and (ii) the theory T is ω-inconsistent, a contradiction. So, $T+\sigma$ must be ω-consistent. ∎

As a corollary, we show that all the Kreiselian sentences are $\mathbb{P}$-provably equivalent to one another, for a given arithmetization of syntax.

Corollary 3.5 ($\mathbb{P} \vdash \kappa \equiv \neg\omega\text{--}\mathtt{Con}_T$): *If κ is a Kreiselian sentence of the* RE *theory T, then $\mathbb{P} \vdash \kappa \leftrightarrow \neg\omega\text{--}\mathtt{Con}_T$.*

Proof: Argue inside $\mathbb{P}$: If κ then $\neg\omega\text{--}\mathtt{Con}_{T+\kappa}$ by Definition 2.9, which implies $\neg\omega\text{--}\mathtt{Con}_T$ by Theorem 3.4 (noting that $\kappa \in \Sigma_3$); therefore, κ implies $\neg\omega\text{--}\mathtt{Con}_T$. Conversely, if $\neg\omega\text{--}\mathtt{Con}_T$ then $\neg\omega\text{--}\mathtt{Con}_{T+\kappa}$ and so κ by Definition 2.9; therefore, $\neg\omega\text{--}\mathtt{Con}_T$ implies κ. ∎

By the $\mathbb{P}$-provable equivalence of κ with $\neg\omega\text{--}\mathtt{Con}_T$, we have the following corollary which is the ω-version of Gödel's Second Incompleteness Theorem, that was first proved by Rosser (*1937*); see also *Boolos 1993*, p. xxxi.

Corollary 3.6 ($\omega\text{--}\mathtt{Con}_T \Longrightarrow \omega\text{--}\mathtt{Con}_{T+\neg\omega\text{--}\mathtt{Con}_T}$): *If the* RE *theory T is ω-consistent, then so is $T+\neg\omega\text{--}\mathtt{Con}_T$.*

In fact, the existence of an unsound ω-consistent theory should have been known from *Rosser 1937*, Theorem 1; notice that if T is ω-consistent, then $T+\neg\omega\text{--}\mathtt{Con}_T$ is unsound and ω-consistent by Corollary 3.6. As noted before Definition 2.9 above, it seems that no other (explicit) proof for the weakness of ω-consistency with respect to soundness, though implicit in *Rosser 1937*, appears in print until *Kreisel 1955*; see also *Lindström 1997*, p. 36, *Isaacson 2011*, Proposition 19, and *Lajevardi and Salehi 2021*, p. 279.

Thus far, we have seen some ω-versions of Lindenbaum's lemma and also Gödel's first and second incompleteness theorems. We do not claim novelty for any of these results;[3]

[3] See e.g. the https://t.ly/GmquO link of the online forum MathOverFlow whose Proposition 1 (due to Emil Jeřábek, saying that 'If $T \supseteq Q$ is ω-consistent, and ϕ is a true Σ_3-sentence, then $T+\phi$ is ω-consistent') is half of our Theorem 2.5.

nevertheless, the following ω-version of Rosser's incompleteness theorem seems to be new.

Theorem 3.7 ($\boldsymbol{\omega}$—$\mathbf{Con}_T \Longrightarrow \exists\rho\in\Pi_3$—$\mathbf{Th}(\mathbb{N})$: $\boldsymbol{\omega}$—$\mathbf{Con}_{T+\rho}\wedge\boldsymbol{\omega}$—$\mathbf{Con}_{T+\neg\rho}$): *If T is an ω-consistent RE theory, then there exists some true Π_3-sentence ρ such that both $T+\rho$ and $T+\neg\rho$ are ω-consistent.*

Proof: By Diagonal Lemma, there exists a Π_3-sentence ρ such that (see Definition 2.7)

$$(\maltese)\quad \mathbb{P}\vdash\rho\leftrightarrow\forall\chi\,[\mho_{T+\neg\rho}(\chi)\rightarrow\exists\xi<\chi\,\mho_{T+\rho}(\xi)].$$

a. We first show that $T+\rho$ is ω-consistent.
   Assume not; then for some (fixed, standard) formula $\boldsymbol{\varphi}(x)$, the Π_2-sentence $\mho_{T+\rho}(\ulcorner\boldsymbol{\varphi}\urcorner)$ is true, so we have Π_2—Th$(\mathbb{N})\vdash\mho_{T+\rho}(\ulcorner\boldsymbol{\varphi}\urcorner)$. Also, by Proposition 2.3, $T+\neg\rho$ is ω-consistent. Thus, $U=T+\neg\rho+\Pi_2$—Th$(\mathbb{N})$ is consistent by Corollary 2.6. Now, by ($\maltese$) we have $U\vdash\exists\chi\,[\mho_{T+\neg\rho}(\chi)\wedge\forall\xi<\chi\neg\mho_{T+\rho}(\xi)]$. Since $U\vdash\mho_{T+\rho}(\ulcorner\boldsymbol{\varphi}\urcorner)$ then we have $U\vdash\exists\chi\leqslant\ulcorner\boldsymbol{\varphi}\urcorner\mho_{T+\neg\rho}(\chi)$. But by the ω-consistency of the theory $T+\neg\rho$, the Σ_2-sentence $\forall\chi\leqslant\ulcorner\boldsymbol{\varphi}\urcorner\neg\mho_{T+\neg\rho}(\chi)$ is true, and so should be Π_2—Th$(\mathbb{N})$-provable. Thus, U is inconsistent; a contradiction. Therefore, $T+\rho$ must be ω-consistent.
b. We now show that $T+\neg\rho$ is ω-consistent.
   If not, then by Proposition 2.3, the theory $T+\rho$ should be ω-consistent, and so the theory $U=T+\rho+\Pi_2$—Th$(\mathbb{N})$ should be consistent by Corollary 2.6. Also, for some formula $\boldsymbol{\varphi}(x)$ we should have Π_2—Th$(\mathbb{N})\vdash\mho_{T+\neg\rho}(\ulcorner\boldsymbol{\varphi}\urcorner)$. Now, by ($\maltese$) we have $U\vdash\exists\xi<\ulcorner\boldsymbol{\varphi}\urcorner\mho_{T+\rho}(\xi)$. But $\forall\xi<\ulcorner\boldsymbol{\varphi}\urcorner\neg\mho_{T+\rho}(\xi)$ is a true Σ_2-sentence by the ω-consistency of the theory $T+\rho$. So, $\forall\xi<\ulcorner\boldsymbol{\varphi}\urcorner\neg\mho_{T+\rho}(\xi)$ is Π_2—Th$(\mathbb{N})$-provable, which implies that U is inconsistent; a contradiction. Therefore, $T+\neg\rho$ must be ω-consistent.

So, both of the theories $T+\rho$ and $T+\neg\rho$ are ω-consistent, whence the Π_3-sentence ρ is true (by the soundness of $\mathbb{P}$). ■

Note that Theorem 3.7 is optimal in a sense, since by Theorem 2.5 for no true Σ_3-sentence σ can the theory $T+\neg\sigma$ be ω-consistent. We end the paper with the observation that Theorem 3.7 can be formalized in $\mathbb{P}$.

Corollary 3.8 ($\boldsymbol{\omega}$—$\mathbf{Con}_T\rightarrow\boldsymbol{\rho}\not\rightarrow\boldsymbol{\omega}$—$\mathbf{Con}_T$): *If ρ is a Π_3-sentence that was constructed for the RE theory T in the Proof of Theorem 3.7 ($\maltese$), then*

(1) $\mathbb{P}\vdash\omega\text{—}\mathtt{Con}_T\longrightarrow\rho\wedge\omega\text{—}\mathtt{Con}_{T+\rho}\wedge\omega\text{—}\mathtt{Con}_{T+\neg\rho}$, *and*
(2) *if T is ω-consistent, then $T\nvdash\rho\rightarrow\omega\text{—}\mathtt{Con}_T$; moreover, $T+\rho+\neg\omega\text{—}\mathtt{Con}_T$ is ω-consistent.*

Proof: For part (2), it suffices to note that since $T+\rho+\neg\omega\!-\!\mathtt{Con}_{T+\rho}$ is ω-consistent by Theorem 3.7 and Corollary 3.6, and $\mathbb{P}\vdash\neg\omega\!-\!\mathtt{Con}_{T+\rho}\leftrightarrow\neg\omega\!-\!\mathtt{Con}_T$ by part (1), then the theory $T+\rho+\neg\omega\!-\!\mathtt{Con}_T$ is ω-consistent as well. ■

4. Conclusions

Since Gödel knew (well before 1931) that (the semantic notion of) soundness *is not* arithmetically definable, while (the syntactic notion of) ω-consistency *is* formalizable (Definition 2.8), then one may guess that this could be his reason for the (much greater) weakness of the latter in comparison with the former. Going deeper into the probably first printed proof of this fact (by Kreisel in 1955) we see that ω-consistency does not imply Σ_3-soundness (Corollary 2.11), though every ω-consistent theory is Π_3-sound (Corollary 2.6). Despite the fact that full soundness implies ω-consistency, we noted that for any natural m there exist Σ_m-sound theories which are not ω-consistent (Theorem 2.12). We also observed the ω-versions of Lindenbaum's Lemma (in Proposition 2.3), Gödel's second incompleteness theorems (in Corollary 3.6), and Rosser's incompleteness theorem (in Theorem 3.7). Not every property of (simple) consistency holds for ω-consistency; for example, the union of a chain of ω-consistent theories is not necessarily ω-consistent (Corollary 3.2), and no ω-consistent theory can be complete unless it is the full True Arithmetic (Proposition 3.1).

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