THE MATHEMATICAL GAZETTE
https://doi.org/10.1080/00255572.2026.2652924

NOTE

A neglected formula in trigonometry: sine and cosine of *x*/2 in terms of sin *x* only

Saeed Salehi

Harish and Bina School of AI and Computer Science, Plaksha University, Mohali, India

Some well-known formulas exist for computing $\cos(\alpha/2)$ and $\sin(\alpha/2)$ for α between 0° and 90°, when $\cos\alpha$ is known (see, e.g. [1, p. 149]):

$$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1+\cos\alpha}{2}} \quad \text{and} \quad \sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1-\cos\alpha}{2}}$$

These can be derived from the double-angle formulas for cosine, namely, $\cos(2\theta) = 2\cos^2\theta - 1$ and $\cos(2\theta) = 1 - 2\sin^2\theta$. Let us write the above half-angle formulas more compactly as (see [2] for a proof without words):

$$\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{\alpha}{2}\right) = \frac{1}{2}\sqrt{2 \pm 2\cos\alpha}$$

A natural question that comes to mind is whether there exist formulas for computing $\cos(\alpha/2)$ and $\sin(\alpha/2)$ in terms of $\sin\alpha$ only. A most obvious answer results from replacing $\cos\alpha$ by $\sqrt{1-\sin^2\alpha}$:

$$\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{\alpha}{2}\right) = \frac{1}{2}\sqrt{2 \pm 2\sqrt{1-\sin^2\alpha}}$$

CONTACT Saeed Salehi saesal@gmail.com Plaksha University, IT City Road, Sector 101A, Mohali, Punjab 140 306, India

But more can be done here, in a prettier way:

$$
\begin{aligned}
&2 \pm 2\sqrt{1-\sin^2\alpha} \\
&\quad = 2 \pm 2\sqrt{(1+\sin\alpha)(1-\sin\alpha)} \\
&\quad = (1+\sin\alpha) + (1-\sin\alpha) \pm 2\sqrt{(1+\sin\alpha)}\sqrt{(1-\sin\alpha)} \\
&\quad = (\sqrt{1+\sin\alpha} \pm \sqrt{1-\sin\alpha})^2
\end{aligned}
$$

So, this seems to be a neglected formula in trigonometry: for every α between 0° and 90°, we have

$$
\boxed{\boxed{\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{\alpha}{2}\right) = \frac{1}{2}\left(\sqrt{1+\sin\alpha} \pm \sqrt{1-\sin\alpha}\right)}}
$$

The peculiar form of the formula can perhaps explain why it has not been (discovered or) recognized as a standard formula for computing the co/sine of half-angles from the sine of the original angle. Unfortunately, I know of no nice geometric demonstration or interpretation of this formula, as it contains the sum/difference of two radicals inside which the sum/difference of 1 and $\sin\alpha$ appear. The whole formula and its proof are highly algebraic; so is the second derivation of the formula that I present now.

Another way of getting this formula is using the double-angle formula for sine: $\sin(2\theta) = 2\cdot\sin\theta\cdot\cos\theta$. So, the product $(\sin\theta\cdot\cos\theta)$ is known: $(1/2)\sin(2\theta)$. Let us compute the sum $(\sin\theta + \cos\theta)$:

$$
\begin{aligned}
&(\sin\theta+\cos\theta)^2 \\
&\quad = (\sin^2\theta+\cos^2\theta) + 2\sin\theta\cos\theta \\
&\quad = 1+\sin(2\theta)
\end{aligned}
$$

So, for $(\sin\theta+\cos\theta) \geqslant 0$, we have $\sin\theta+\cos\theta = \sqrt{1+\sin(2\theta)}$. Thus $\sin\theta$ and $\cos\theta$ are the roots of the following quadratic equation:

$$
x^2 - \sqrt{1+\sin(2\theta)}\,x + \frac{1}{2}\sin(2\theta) = 0
$$

Now, by using the delta formula, $\frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac})$, we get the following formula for $\cos\theta \geqslant \sin\theta$:

$$\boxed{\begin{matrix}\cos\\ \sin\end{matrix}(\theta) = \frac{1}{2}\left(\sqrt{1+\sin(2\theta)} \pm \sqrt{1-\sin(2\theta)}\right)}$$

In particular, for θ in $[0^\circ, 45^\circ]$, we obtain our seemingly neglected trigonometric formula. Finally, let us give another formula for computing the co/sine of $\alpha/2$ in terms of *both* $\sin\alpha$ and $\cos\alpha$ as an exercise, with a big apology to the senior readers:

Exercise. Show that for every α between 0° and 90°, we have

$$\boxed{\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{\alpha}{2}\right) = \frac{\sin\alpha}{\sqrt{2 \mp 2\cos\alpha}}}$$

Author contributions

CRediT: **Saeed Salehi:** Conceptualization, Investigation, Methodology, Supervision, Writing – original draft, Writing – review & editing

Disclosure statement

No potential conflict of interest was reported by the author(s).

ORCID

Saeed Salehi http://orcid.org/0000-0001-6961-5749

References

[1] Gelfand I. M., Saul M. (2001). Trigonometry. Birkhäuser, Boston (MA, USA).
[2] Salehi S. (2026). Proofs without words: Sine and cosine of half-angles. p. 1. DOI: 10.2139/ssrn.6101326.