

MR3201831 (Review) 03F03 03F45 03F50**Dean, Walter** (4-WARW-Q)**Montague's paradox, informal provability, and explicit modal logic. (English summary)***Notre Dame J. Form. Log.* **55** (2014), no. 2, 157–196.

This long paper, which is a continuation of the author's Ph.D. thesis [*Epistemic paradox and explicit modal logic*, City Univ. New York, 2010], concerns the interpretation of Montague's Paradox, which is taken to be the following statement: Any theory whose language contains the language of arithmetic $(0, S, +, \times)$ and a unary predicate P , which extends Robinson's Arithmetic (Q) and satisfies the axiom $P(\ulcorner \varphi \urcorner) \rightarrow \varphi$ and the rule $\vdash \varphi / \vdash P(\ulcorner \varphi \urcorner)$, is inconsistent.

"The main thesis of this paper" is that "the significance of Montague's paradox is most readily appreciated if we understand $P(x)$ as expressing some form of mathematical provability". The author suggests that "by interpreting $P(x)$ as expressing provability, we not only gain some insight into what distinguishes the significance of [Montague's paradox] from formally similar inconsistency results (such as Tarski's formalization of the liar paradox), but also open new avenues for resolving the underlying conceptual tension which the result might be taken to highlight on this interpretation". Then three notions of mathematical provability are distinguished: (1) informal provability, (2) formal provability, and (3) constructive provability. The author argues that "not only do variants of Montague's paradox arise for each of the interpretations just considered, but also that a common resolution is available in each case". *Saeed Salehi*

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