MR3109604 (Review) 03F65 03F50 13L05
Schuster, Peter [Schuster, Peter M.] (4-LEED-PM)
Induction in algebra: a first case study. (English summary)
Log. Methods Comput. Sci. **9** (2013), *no. 3,* 3:20, 19 *pp.*

Consider a classical theorem in ring theory stating that "every nonconstant coefficient of an invertible polynomial is nilpotent". Symbolically, for any $\overline{a}$ and $\overline{b}$, under the assumption (A) $\forall x\big[(\sum_{i=0}^{n} a_i x^i)(\sum_{i=0}^{m} b_i x^i) = 1\big]$ we can conclude that (B) $\bigwedge_{i=1}^{n} \exists e_i (a_i^{u_i} = 0)$ holds. A classical proof argues as follows: for any prime ideal P of the ring $\mathcal{R}$, the quotient ring $\mathcal{R}/P$ is an integral domain, so (A) implies that $\bigwedge_{i=1}^{n}(a_i = 0)$ (in the quotient integral domain $\mathcal{R}/P$). Hence, for any prime ideal P of $\mathcal{R}$ the assumption (A) implies (C) $\bigwedge_{i=1}^{n}(a_i \in P)$. Now, by a variant of Krull's Lemma, which is normally deduced from Zorn's Lemma, (C) implies (B). The author of the paper under review argues that this proof, being highly non-constructive, "loses the computational information"; in particular, the proof of (A)$\to$(B) falls short of being an algorithm for computing exponents e_i ($1 \le i \le n$) under which the nilpotent coefficients a_i vanish.

The author provides a constructive proof by first turning the indirect proof of (A)$\to$(B), which uses Zorn's Lemma, into a direct deduction from the principle of Open Induction, which, in a nutshell, is "transfinite induction for subsets of a directed-complete partial order that are open with respect to the Scott topology", and then transforming the latter into a constructive proof of (A)$\to$(B) by induction over a finite poset. *Saeed Salehi*

References

1. Peter Aczel. The type theoretic interpretation of constructive set theory. In *Logic Colloquium '77 (Proc. Conf., Wrocław, 977)*, volume 96 of *Stud. Logic Foundations Math.*, pages 55–66. North-Holland, Amsterdam, 1978. MR0519801 (80f:03068)
2. Peter Aczel. The type theoretic interpretation of constructive set theory: choice principles. In *The L. E. J. Brouwer Centenary Symposium (Noordwijkerhout, 1981)*, volume 110 of *Stud. Logic Found. Math.*, pages 1–40. North-Holland, Amsterdam, 1982. MR0717236 (85g:03085)
3. Peter Aczel. The type theoretic interpretation of constructive set theory: inductive definitions. In *Logic, methodology and philosophy of science, VII (Salzburg, 1983)*, volume 114 of *Stud. Logic Found. Math.*, pages 17–49. North-Holland, Amsterdam, 1986. MR0874778 (88a:03149)
4. Michael F. Atiyah and Ian G. Macdonald. *Introduction to Commutative Algebra.* Addison-Wesley Publishing Co., 1969. MR0242802 (39 #4129)
5. B. Banaschewski and J. J. C. Vermeulen. Polynomials and radical ideals. *J. Pure Appl. Algebra*, 113(3):219–227, 1996. MR1417392 (98c:13004)
6. Bernhard Banaschewski. Radical ideals and coherent frames. *Comment. Math. Univ. Carolin.*, 37(2):349–370, 1996. MR1399006 (97g:13002)
7. John L. Bell. Zorn's lemma, and complete Boolean algebras in intuitionistic type theories. *J. Symbolic Logic*, 62(4):1265–1279, 1997. MR1617953 (99g:03062)
8. Ulrich Berger. A computational interpretation of open induction. In F. Titsworth, editor, *Proceedings of the Ninetenth Annual IEEE Symposium on Logic in Computer Science*, pages 326–334. IEEE Computer Society, 2004.
9. Douglas S. Bridges, Prime and maximal ideals in constructive ring theory. *Commun.*

Algebra, 29:2787–2803, 2001. MR1848382 (2002e:03101)
10. Paul M. Cohn. *Universal Algebra*. Harper & Row Publishers, New York, 1965. MR0175948 (31 #224)
11. Thierry Coquand. Constructive topology and combinatorics. In *Constructivity in computer science (San Antonio, TX, 1991)*, volume 613 of *Lecture Notes in Comput. Sci.*, pages 159–164. Springer, Berlin, 1992. MR1250673
12. Thierry Coquand. Space of valuations. *Ann. Pure Appl. Logic*, 157:97–109, 2009. MR2499701 (2010k:03056)
13. Thierry Coquand and Henri Lombardi. A logical approach to abstract algebra. *Math. Struct. in Comput. Science*, 16:885–900, 2006. MR2268347 (2007j:03089)
14. Thierry Coquand and Henrik Persson. Gröbner bases in type theory. In *Types for proofs and programs (Irsee, 1998)*, volume 1657 of *Lecture Notes in Comput. Sci.*, pages 33–46. Springer, Berlin, 1999. MR1853594
15. Thierry Coquand and Henrik Persson. Valuations and Dedekind's Prague theorem. *J. Pure Appl. Algebra*, 155(2–3):121–129, 2001. MR1801410 (2001m:13032)
16. Michel Coste, Henri Lombardi, and Marie-Françoise Roy. Dynamical method in algebra: Effective Nullstellensätze. *Ann. Pure Appl. Logic*, 111(3):203–256, 2001. MR1848137 (2003d:03104)
17. Laura Crosilla and Peter Schuster. Finite Methods in Mathematical Practice. In G. Link and M. Detlefsen, editors, *Formalism and Beyond*, Mathematical Logic. Ontos, Heusenstamm, 201x.
18. Jean Della Dora, Claire Dicrescenzo, and Dominique Duval. About a new method for computing in algebraic number fields. In *European Conference on Computer Algebra (2)*, pages 289–290, 1985. 19 Harvey Friedman. Set theoretic foundations for constructive analysis. *Ann. of Math. (2)*, 105(1):1–28, 1977. MR0434784 (55 #7748)
19. Matthew Hendtlass and Peter Schuster. A direct proof of Wiener's theorem. In S. B. Cooper, A. Dawar, and B. Löwe, editors, *How the World Computes. Turing Centenary Conference and Eighth Conference on Computability in Europe*, volume 7318 of *Lect. Notes Comput. Sci.*, pages 294–303, Berlin and Heidelberg, 2012. Springer. Proceedings, CiE 2012, Cambridge, UK, June 2012. MR2983692
20. Simon Huber and Peter Schuster. Maximalprinzipien und Induktionsbeweise. Technical report, University of Leeds, 2013. In preparation.
21. Hajime Ishihara. A note on the Gödel-Gentzen translation. *MLQ Math. Log. Q.*, 46(1):135–137, 2000. MR1736658 (2000k:03029)
22. Carl Jacobsson and das Löfwall. Standard bases for general coefficient rings and a new constructive proof of Hilbert's basis theorem. *J. Symb. Comput.*, 12(3):337–372, 1991. MR1128249 (92j:13027)
23. Peter T. Johnstone. *Stone Spaces*. Number 3 in Cambridge Studies in Advanced Mathematics. Cambridge etc.: Cambridge University Press, 1982. MR0698074 (85f:54002)
24. André Joyal. Les théoremes de Chevalley-Tarski et remarques sur l'algèbre constructive. *Cah. Topol. Géom. Différ. Catég.*, 16:256–258, 1976.
25. Irving Kaplansky. *Commutative Rings*. The University of Chicago Press, Chicago and London, 1974. Revised edition. MR0345945 (49 #10674)
26. Wolfgang Krull. *Idealtheorie*. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 4, no. 3. Springer, Berlin, 1935. MR0229623 (37 #5197)
27. Henri Lombardi. Dimension de Krull, Nullstellensätze et évaluation dynamique. *Math. Zeitschrift*, 242:23–46, 2002. MR1985448 (2004f:13013)
28. Henri Lombardi. Hidden constructions in abstract algebra. I. Integral dependance. *J. Pure Appl. Algebra*, 167:259–267, 2002. MR1874544 (2002i:13004)

29. Henri Lombardi. Algfèbre dynamique, espaces topologiques sans points et programme de Hilbert. *Ann. Pure Appl. Logic*, 137:256–290, 2006. MR2182105 (2006i:03103)
30. Henri Lombardi and Claude Quitté. *Algèbre commutative. Méthodes constructives. Modules projectifs de type fini*. Calvage & Mounet, Paris, 2012.
31. Maria Emilia Maietti. A minimalist two-level foundation for constructive mathematics. *Ann. Pure Appl. Logic*, 160(3):319–354, 2009. MR2555783 (2011a:03073)
32. Maria Emilia Maietti and Giovanni Sambin. Toward a minimalist foundation for constructive mathematics. In L. Crosilla and P. Schuster, editors, *From Sets and Types to Topology and Analysis*, volume 48 of *Oxford Logic Guides*, pages 91–114. Oxford: Oxford University Press, 2005. MR2188638 (2006f:03103)
33. Per Martin-Löf. *Intuitionistic type theory*, volume 1 of *Studies in Proof Theory. Lecture Notes*. Bibliopolis, Naples, 1984. Notes by Giovanni Sambin. MR0769301 (86j:03005)
34. Ray Mines, Fred Richman, and Wim Ruitenburg. *A Course in Constructive Algebra*. Springer, New York, 1988. Universitext. MR0919949 (89d:03066)
35. Sara Negri. Contraction-free sequent calculi for geometric theories with an application to Barr's theorem. *Arch. Math. Logic*, 42(4):389–401, 2003. MR2018089 (2004i:03091)
36. Douglas G. Northcott. *Ideal Theory*. Cambridge University Press, 1953. MR0058575 (15,390f)
37. Erik Palmgren. An intuitionistic axiomatisation of real closed fields. *MLQ Math. Log. Q.*, 48(2):297–299, 2002. MR1883244 (2003b:03089)
38. Hervé Perdry. Strongly Noetherian rings and constructive ideal theory. *J. Symb. Comput.*, 37(4):511–535, 2004. MR2093449 (2006c:13026)
39. Hervé Perdry and Peter Schuster. Noetherian orders. *Math. Structures Comput. Sci.*, 21:111–124, 2011. MR2763080 (2012d:06007)
40. Henrik Persson. An application of the constructive spectrum of a ring. In *Type Theory and the Integrated Logic of Programs*. Chalmers University and University of Götebarg, 1999. PhD thesis.
41. Jean-Claude Raoult. Proving open properties by induction. *Inform. Process. Lett.*, 29(1):19–23, 1988. MR0974194 (90k:04001)
42. Fed Richman. Nontrivial uses of trivial rings. *Proc. Amer. Math. Soc.*, 103(4):1012–1014, 1988. MR0954974 (89g:13003)
43. Giovanni Sambin. Intuitionistic formal spasea—a first communication. In *Mathematical Logic and its Applications, Proc. Adv. Internat. Summer School Conf., Druzhba, Bulgaria, 1986*, pages 187–204. Plenum, 1987. MR0945195 (89f:03056)
44. Giovanni Sambin. Steps towards a dynamic constructivism. In P. Gärdenfors et al., editor, *In the Scope of Logic, Methodology and Philosophy of Science*, volume 315 of *Synthese Library*, pages 263–286, Dordrecht, 2002. Kluwer. 11th International Congress of Logic, Methodology and Philosophy of Science. Krakow, Poland, August 1999.
45. Giovanni Sambin. Some points in formal topology. *Theoret. Comput. Sci.*, 305(1–3):347–408, 2003. MR2013578 (2004m:03229)
46. Giovanni Sambin. Real and ideal in constructive mathematics. In *Epistemology versus ontology*, volume 27 of *Log. Epistemol. Unity Set.*, pages 69–85. Springer, Dordrecht, 2012. MR2962712
47. Peter Schuster. Induction in algebra: a first case study. In *2012 27th Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 581–585. IEEE Computer Society Publications, 2012, Proceedings, LICS 2012, Dubrovnik, Croatia, June 2012. MR3051359

48. Viggo Stoltenberg-Hansen and John V. Tucker. Computable rings and fields. In *Handbook of computability theory*, volume 140 of *Stud. Logic Found. Math.*, pages 363–447. North-Holland, Amsterdam, 1999. MR1720739 (2000g:03100)
49. Pedro Francisco Valencia Vizcaíno. *Some Uses of Cut Elimination*. Phd thesis, University of Leeds, 2013.
50. Ihsen Yengui. Making the use of maximal ideals constructive, *Theoret. Comput. Sci.*, 392:174–178, 2008. MR2394992 (2009a:13012)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.