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**On the decidability of semigroup freeness.** **(English summary)**
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The freeness problem considered in this paper is the following: Given a semigroup $S$ with a recursive underlying set and a finite subset $X \subseteq S$, decide whether $X$ is a code for $S$ or not, where a code is a subset $X \subseteq S$ such that no element of $S$ can have more than one factorization over $X$; or, in other words, for any $m, n \geq 1$ and any elements $x_1, \dots, x_n, y_1, \dots, y_m \in X$, if $x_1 \cdots x_n = y_1 \cdots y_m$ then $m = n$ and $x_i = y_i$ for any $1 \leq i \leq n$. The freeness problem over a semigroup $S$ is denoted by Free$[S]$, and Free$(k)[S]$ denotes the following problem: Given a $k$-element subset $X \subseteq S$, decide whether $X$ is a code for $S$ or not.

The paper is a survey of this problem for various semigroups, and includes some new results. In section 2 the authors study problems related with matrix torsion, where a square matrix $M$ is called torsion if there are $p, q \geq 1$ such that $M^p = M^{p+q}$; $M$ is torsion if and only if the singleton $\{M\}$ is not a code under matrix multiplication. In section 3 it is proved that any subset $X \subseteq S$ with cardinality greater than 1 is not a code if and only if the elements of $X$ satisfy a nontrivial balanced equation. One particular consequence of this is that for $d \geq 1$, Free$[\mathbb{Q}^{d\times d}]$ reduces to Free$[\mathbb{Z}^{d\times d}]$ where $A^{d\times d}$, for a set $A$, is the set of $d \times d$ matrices over $A$. In section 4 it is shown that Free$[\mathrm{GL}(2,\mathbb{Z})]$ is decidable and for every finite alphabet $\Sigma$, Free$[\mathrm{FG}(\Sigma)]$ is decidable in polynomial time, where $\mathrm{GL}(2,\mathbb{Z})$ is the set of invertible $2 \times 2$ matrices over $\mathbb{Z}$ and $\mathrm{FG}(\Sigma)$ is the free group over $\Sigma$. In section 5 the authors show that for $k \geq 1$ the decidability of the problem Free$(k+1)[S]$ does not necessarily imply the decidability of the problem Free$(k)[S]$. It is also shown that for any semigroup $S$ with a computable operation, either Free$(k)[S]$ is decidable for every $k \geq 2$, or Free$(k)[S]$ is undecidable for infinitely many $k$'s. In section 6 the open problem Free$[\mathbb{N}^{2\times 2}]$ is studied, and in section 7 the undecidability of Free$(k)[\mathbb{W} \times \mathbb{W}]$ and of Free$(k)[\mathbb{N}^{3\times 3}]$ for every $k \geq 13$, where $\mathbb{W} = \{0,1\}^*$, is shown. These are some new results of the paper. Finally, in section 8 undecidability of the following problems for every $h \in \mathbb{N}$ is shown: Free$(7+h)[\mathbb{N}^{6\times 6}]$, Free$(5+h)[\mathbb{N}^{9\times 9}]$, Free$(4+h)[\mathbb{N}^{12\times 12}]$, Free$(3+h)[\mathbb{N}^{18\times 18}]$, and Free$(2+h)[\mathbb{N}^{36\times 36}]$.

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