MR2838322 (2012k:68134) 68Q30 03D10 03F30
Ibuka, Shingo (J-KOBEE-CE);
Kikuchi, Makoto [Kikuchi, Makoto[1]] (J-KOBEE-IF);
Kikyo, Hirotaka (J-KOBEE-IF)

Kolmogorov complexity and characteristic constants of formal theories of arithmetic. (English summary)

MLQ Math. Log. Q. **57** (2011), *no. 5,* 470–473.

For a fixed universal Turing machine (UTM) Φ which can enumerate all Turing machines with inputs, the Kolmogorov complexity of a natural number x is the least natural number y such that $\Phi_y(0)\downarrow x$, i.e., the Turing machine with code y halts with input 0 and outputs x; in notation $K^\Phi(x)=y$. The characteristic constant (or Chaitin constant) c_T^Φ of a formal theory T is the least natural number y such that for any natural number x, $T\nvdash K^\Phi(x)>y$. The Raatikainen constant r_T^Φ of a formal theory T is the least natural number i such that the Turing machine with code i on input 0 does not terminate ($\Phi_i(0)\uparrow$) and T cannot prove this fact; that is, $T\nvdash \Phi_i(0)\uparrow$. A very brief history of these constants is that G. J. Chaitin [J. Assoc. Comput. Mach. **21** (1974), 403–424; MR0455537 (56 #13775)] proved that for any recursively axiomatized consistent formal system T there exists a number c_T such that for no x, can T prove $K(x)>c_T$, though there are some x's such that $K(x)>c_T$ is true. Then this constant c_T was taken to be the complexity of the theory T and it was claimed that T cannot prove sentences with complexity greater than c_T. P. Raatikainen [J. Philos. Logic **27** (1998), no. 6, 569–586; MR1663393 (2000b:68102)], criticizing the received interpretation of Chaitin's constant in the logic literature, showed that for some specific choice of a UTM the constant c_T can be zero; thus, as the authors of the current paper argue, $c_S=c_T$ is always possible for two theories S and T. Whence the constant c_T cannot measure any complexity of a theory T.

The main result of the current paper is that for two theories S and T extending Peano arithmetic, the following are equivalent:

(1) There exists a Π_1-sentence which is provable in T but not in S, or in notation, $\Pi_1(T)\nsubseteq\Pi_1(S)$;

(2) $r_S=r_T$ and $c_S<c_T$ for some UTM;

(3) $r_S<r_T$ and $c_S<c_T$ for some UTM;

(4) $r_S<r_T$ and $c_S=c_T$ for some UTM.

Whence, the authors' statement in the abstract "We prove the following are equivalent: $c_S\neq c_T$ for some universal Turing machine, $r_S\neq r_T$ for some universal Turing machine, and T proves some Π_1-sentence which S cannot prove" is misleading, since the condition "$c_S\neq c_T$ for some UTM" is not equivalent to the condition "$r_T\neq r_S$ for some UTM". Instead "$\Pi_1(T)\nsubseteq\Pi_1(S)$" is shown to be equivalent to "$r_S<r_T$ and $c_S\leqslant c_T$ for some UTM" and to "$r_S\leqslant r_T$ and $c_S<c_T$ for some UTM".

It is also proved that for any UTM Φ and any natural number n there exists a UTM Ψ such that $K^\Phi=K^\Psi$, $r_T^\Phi=r_T^\Psi$ and $r_T^\Psi+n<c_T^\Psi$. *Saeed Salehi*

References

1. G. J. Chaitin, Information-theoretic limitations of formal systems, J. ACM **21**, 403–424 (1974). MR0455537 (56 #13775)
2. S. Ibuka, M. Kikuchi, and H. Kikyo, On Characteristic Constants Defined by Kolmogorov Complexity, in: Logic, Language, Information and Computation, 15th

International Workshop, WoLLIC 2008 Edinburgh, UK, July 1–4, 2008, edited by W. Hodges and R. de Queiroz, Lecture Notes in Artificial Intelligence Vol. 5110 (Springer-Verlag, 2008), pp. 218–225. MR2500394 (2010h:68076)

3. M. van Lambalgen, Algorithmic information theory, J. Symb. Log. **54**, 1389–1400 (1989). MR1026605 (92b:68037)
4. M. Li and P. Vitanyi, An Introduction to Kolmogorov Complexity and Its Applications, second ed. (Springer-Verlag, 1997). MR1438307 (97k:68086)
5. P. Raatikainen, On interpreting Chaitin's incompleteness theorem, J. Philos. Log. **27** (6), 569–586 (1998). MR1663393 (2000b:68102)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.