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Partial monoids: associativity and confluence. (English summary)
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A partial monoid (or a pre-monoid) is a non-empty set P with a partial binary function $\circ\colon D \to P$ and an identity element $1_P \in P$ such that $D \subseteq P \times P$ and for every $x \in P$ we have $(x, 1_P), (1_P, x) \in D$ and $x \circ 1_P = x = 1_P \circ x$, and for every $x, y, z \in P$ we have $(x, y), (x \circ y, z) \in D$ if and only if $(y, z), (x, y \circ z) \in D$, and in either case $x \circ (y \circ z) = (x \circ y) \circ z$. Such a partial monoid can be embedded into the free monoid P^* with the concatenation operation. Let $i_P\colon P \hookrightarrow P^*$ be that embedding, and define the semi-Thue system R_P to be

$$\{\langle i_P(x)i_P(y), i_P(x \circ y)\rangle \,|\, (x, y) \in D\} \cup \{\langle i_P(1_P), \varepsilon\rangle\}.$$

The authors note that if P is catenary associative, then R_P is confluent; but the converse does not hold. The partial monoid P is called catenary associative when for all $x, y, z \in P$ if $y \neq 1_P$ and $(x, y) \in D$ and $(y, z) \in D$ then $(x \circ y, z) \in D$ (thus also $(x, y \circ z) \in D$). The set of irreducible elements of P^* (under R_P) is denoted by $\mathrm{Irr}(P)$. The operation $\star$ is defined on $\mathrm{Irr}(P)$ by letting $u \star v$ be the left standard reduction set of uv. The operation $\star$ is associative modulo R_P. The main result of the paper is that the operation $\star$ is associative if and only if R_P is confluent. Hence, in that case $\mathrm{Irr}(P)$ is isomorphic to $P^*/\Leftrightarrow^*_{R_P}$, and in the case that P is a total monoid, then both of them are isomorphic to P itself.

The paper contains most of the necessary preliminaries from the theory of rewriting systems, and the proofs are somehow straightforward; the authors note that most of the proofs of lemmas are omitted, since they are free of technical difficulties. *Saeed Salehi*