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A constructive version of Birkhoff's theorem. (English summary)
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The author presents two theorems as natural constructive counterparts to Birkhoff's classical theorem (that a class of algebras is axiomatized by a set of identities if and only if it is closed under subalgebras, homomorphic images, and products). An algebra G is called generic for a class $\mathcal{C}$ of algebras when any identity that holds in G also holds in any member of $\mathcal{C}$. The first constructive theorem states: if $\mathcal{C}$ contains a generic member G, and

(i) is closed under homomorphic images,
(ii) contains those algebras whose finitely-generated subalgebras are in $\mathcal{C}$, and
(iii) contains every power of G,

then it is axiomatized by the set of G-identities.

The author notes that many natural classes of algebras, like commutative rings, fields and Boolean algebras, contain a generic member.

A set-indexed family $\{G_i\}$ of algebras of the class $\mathcal{C}$ is called a generic family when any identity which holds in every G_i also holds in every member of $\mathcal{C}$. By impredicative methods and using the powerset and full separation one can prove that every class that is closed under isomorphisms and subalgebras contains a generic family. The second constructive theorem states that: if $\mathcal{C}$ contains a generic family $\{G_i\}$ and is closed under

(i) homomorphic images,
(ii) subalgebras,
(iii) inductive limits, and
(iv) products,

then it is axiomatized by $\prod G_i$-identities.

Reviewed by *Saeed Salehi*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.