On Herbrand Consistency of Bounded Arithmetics

Saeed Salehi

University of Tabriz

http://SaeedSalehi.ir/

Logic Colloquium 2011

June 2011 – Barcelona, Spain

Glad To Be Back

My Last Talk in Logic Colloquium:

2001, Vienna, Austria
"Unprovability of Herbrand Consistency in Weak Arithmetics"

LC'2000, Paris, France – "A Generalized Realizability for Constructive Arithmetics"

LC'1999, Utrecht, Holland – "Intuitionistic Axiomatization of End-Extension Kripke Models"

On Herbrand Consistency of Bounded Arithmetics Logic Colloquium 2011 (2/16)

Bounded Quantifiers

- All $\exists x$ are in the form $\exists x \leqslant t$
- All $\forall y$ are in the form $\forall y \leqslant s$

t, s are $\cdots$ terms

Bounded Formula: all quantifiers are bounded.

▶ Relations definable by bounded formulas are

- Decidable
- Primitive Recursive
- Recognizable in Linear Space [LinSpace $\equiv$ Space $\in \mathcal{O}(n)$]
- Recognizable in the Linear Time Hierarchy

Peano Arithmetic

Robinson's Arithmetic Q:

- $\mathrm{S}(x) = \mathrm{S}(y) \Rightarrow x = y$
- $x + 0 = x$
- $x \cdot 0 = 0$
- $x \leqslant y \iff \exists z(z + x = y)$
- $\mathrm{S}(x) \neq 0$
- $x + \mathrm{S}(y) = \mathrm{S}(x + y)$
- $x \cdot \mathrm{S}(y) = (x \cdot y) + x$
- $x \neq 0 \Rightarrow \exists y[x = \mathrm{S}(y)]$

Plus the Induction Axioms:

$$\varphi(0) \wedge \forall x[\varphi(x) \to \varphi(\mathrm{S}(x))] \Longrightarrow \forall y \varphi(y)$$

Bounded Arithmetic

Definition

Q + Induction Axiom for Bounded Formulas $=$ $\mathrm{I}\Delta_0$

Theorem (*R.J. Parikh 1971*)

$\mathrm{I}\Delta_0 \vdash \forall\overline{x}\exists y\ \eta(\overline{x},y)\ \&\ \eta\in\Delta_0 \Longrightarrow \mathrm{I}\Delta_0 \vdash \forall\overline{x}\ \exists y\leqslant t(\overline{x})\ \eta(\overline{x},y)$

$t-$*term*

Provably Recursive Functions of $\mathrm{I}\Delta_0$ are Polynomially Bounded

$$\mathrm{I}\Delta_0 \vdash \forall\overline{x}\exists y\ \underbrace{\eta(\overline{x},y)}_{\Delta_0} \Longrightarrow \mathrm{I}\Delta_0 \vdash \forall\overline{x}\,\underbrace{\exists y\leqslant t(\overline{x})\eta(\overline{x},y)}_{\Delta_0}$$

Why Bounded Arithmetic ?

$x \mid y \equiv \exists z(x \cdot z = y)$ $\qquad$ $\texttt{Prime}(x) \equiv \forall y(y \mid x \Rightarrow y = 1 \vee y = x)$

PA=Peano Arithmetic $\qquad$ $\mathbf{PA} \vdash \forall x \exists y \Big(y > x \wedge \texttt{Prime}(y) \Big)$

Open Problem: $\mathrm{I}\Delta_0 \vdash^{?} \forall x \exists y \Big(y > x \wedge \texttt{Prime}(y) \Big)$

$\mathrm{Exp} = \forall x \exists y [y = 2^x]$ $\qquad$ “ $y = 2^x$ ” $\in \Delta_0$

$\mathrm{EA} = \mathrm{I}\Delta_0 + \mathrm{Exp}$ $\qquad$ $\mathbf{EA} \vdash \forall x \exists y \Big(y > x \wedge \texttt{Prime}(y) \Big)$

Elementary Arithmetic

More Bounded Arithmetic

Definition

$$\begin{cases} \omega_0(x) = x^2 \\ \omega_{n+1}(x) = 2^{\omega_n(\log x)} \end{cases} \qquad \omega_1(x) = 2^{\log x \cdot \log x} \sim x^{\log x}$$

$$\omega_m(x) = \exp^m([\log^m x]^2) \qquad f^m(x) = \underbrace{f \dots f}_{m-\text{times}}(x)$$

$$\text{polynomial}(x) \ll \omega_1(x) \ll \omega_2(x) \ll \cdots \ll 2^x$$

Definition

$$\Omega_m = \forall x \exists y [y = \omega_m(x)] \qquad \text{“ } y = \omega_m(x) \text{ ”} \in \Delta_0$$

$$\mathrm{I}\Delta_0 \subsetneqq \mathrm{I}\Delta_0 + \Omega_1 \subsetneqq \mathrm{I}\Delta_0 + \Omega_2 \subsetneqq \cdots \subsetneqq \mathrm{I}\Delta_0 + \bigwedge_j \Omega_j \subsetneqq \mathrm{I}\Delta_0 + \mathrm{Exp}$$

Unprovabilty of Consistency

$$\mathcal{C}on(\mathrm{T}) \;=\; \text{“ T is consistent ”} \;=\forall z \neg \underbrace{\mathtt{Proof}_{\mathrm{T}}}_{\Delta_0}(z, \ulcorner 0=1 \urcorner) \in \Pi_1$$

Gödel's Second Incompleteness Theorem

$\mathrm{PA} \nvdash \mathcal{C}on(\mathrm{PA})$ $\qquad$ $\mathrm{ZFC} \vdash \mathcal{C}on(\mathrm{PA})$

$\mathrm{I}\Delta_0 \nvdash \mathcal{C}on(\mathrm{I}\Delta_0)$ $\qquad$ $\mathrm{PA} \vdash \mathcal{C}on(\mathrm{I}\Delta_0)$

But $\mathrm{I}\Delta_0 + \mathrm{Exp} \nvdash \mathcal{C}on(\mathrm{I}\Delta_0)$! $\qquad$ How $\mathrm{I}\Delta_0 + \mathrm{Exp} \supsetneqq^{\Pi_1} \mathrm{I}\Delta_0$?

Open Problem: Π_1-Separating the Hierarchy $\{\mathrm{I}\Delta_0 + \Omega_m\}_m$

Herbrand Consistency 1

- Skolemizing: $\exists y \rightsquigarrow$ eliminate $\exists$ & $[\mathfrak{f}(\overline{x}) \hookleftarrow y]$ $\mathfrak{f}$ new symbol
  $\overline{x}$ all the universal variables before y

then eliminating the remaining $\forall$ quantifiers

Examples:

- $\forall x \exists y\, \varphi(x,y) \stackrel{\text{Sk}}{\longrightarrow} \varphi(x, \mathfrak{f}(x))$
- $\exists y \forall u \exists z\, \varphi(y,u,z) \stackrel{\text{Sk}}{\longrightarrow} \varphi\left(\mathfrak{c}, u, \mathfrak{f}(u)\right)$

- T is Consistent $\Longleftrightarrow$ T^{Sk} is Consistent
  `First-Order` $\leftrightsquigarrow$ `Propositional`

Herbrand Consistency 2

Definition

Herbrand Consistency of T = Propositional Satisfiability of every finite set of (Skolem) instances of T

$$\mathrm{I}\Delta_0 + \mathrm{SupExp} \vdash \mathcal{HC}on(\mathrm{T}) \longleftrightarrow \mathcal{C}on(\mathrm{T})$$

$$\mathrm{I}\Delta_0 + \mathrm{Exp} \nvdash \mathcal{HC}on(\mathrm{T}) \longleftrightarrow \mathcal{C}on(\mathrm{T})$$

$\mathrm{I}\Delta_0 + \mathrm{Exp} \vdash \mathcal{HC}on(\mathrm{I}\Delta_0)$

Presumably ... $\mathrm{I}\Delta_0 \nvdash \mathcal{HC}on(\mathrm{I}\Delta_0)$

Unprovabilty of Herbrand Consistency 1

Theorem (*Z. Adamowicz 2001,2002*)

$\mathrm{I}\Delta_0 + \Omega_\mathrm{m} \nvdash \mathcal{HC}on(\mathrm{I}\Delta_0 + \Omega_\mathrm{m})$ for $\mathrm{m} \geqslant 2$.

Theorem (*S. Salehi 2002*)

$\mathrm{I}\Delta_0 + \Omega_1 \nvdash \mathcal{HC}on(\mathrm{I}\Delta_0 + \Omega_1)$.

Theorem (*D.E. Willard 2002*)

$\mathrm{I}\Delta_0 \nvdash \mathcal{HC}on(\mathrm{I}\Delta_0 + \Omega_0)$. $\Omega_0 = \forall x \exists y [y = x^2]$

Theorem (*L.A. Kołodziejczyk 2006*)

$\bigcup_n(\mathrm{I}\Delta_0 + \Omega_\mathrm{n}) \nvdash \mathcal{HC}on(\mathrm{I}\Delta_0 + \Omega_\mathrm{m})$ for $\mathrm{m} \geqslant 1$.

Unprovabilty of Herbrand Consistency 2

Theorem (*Z. Adamowicz 1996*)

$\mathrm{I}\Delta_0 + \Omega_1 \nvdash \mathcal{T}ableau\mathfrak{Con}(\mathrm{I}\Delta_0 + \Omega_1)$.

Theorem (*S. Salehi 2002*)

$U \nvdash \mathcal{H}\mathfrak{Con}(U)$. $\qquad U \in \Pi_2(\mathrm{I}\Delta_0)$

Theorem (*D.E. Willard 2002*)

$\mathrm{I}\Delta_0 \nvdash \mathcal{T}ableau\mathfrak{Con}(\mathrm{I}\Delta_0)$. $\qquad V \nvdash \mathcal{H}\mathfrak{Con}(V)$ for some $V \in \Pi_1(\mathrm{I}\Delta_0)$.

Theorem (*L.A. Kołodziejczyk 2006*)

$\mathrm{I}\Delta_0 + \bigwedge_j \Omega_j \nvdash \mathcal{H}\mathfrak{Con}(T)$. $\qquad T \subseteq_{\text{finite}} \mathrm{I}\Delta_0 + \Omega_1$

Π_1-Separation ?

- So, $\mathrm{I}\Delta_0 + \mathrm{Exp}$ is NOT Π_1-conservative over $\mathrm{I}\Delta_0$ and $\mathcal{HC}on$ can Π_1-separate them.
- $\mathrm{I}\Delta_0 + \mathrm{Exp}$ is NOT Π_1-conservative over even $\mathrm{I}\Delta_0 + \bigwedge_j \Omega_j$ but can $\mathcal{HC}on$ Π_1-separate them?
- However, $\mathcal{HC}on$ <u>cannot</u> Π_1-separate $\mathrm{I}\Delta_0 + \bigwedge_j \Omega_j$ from $\mathrm{I}\Delta_0$!

On Herbrand Consistency of Bounded Arithmetics Logic Colloquium 2011 (13/16)

New Results 1

Theorem (*S. Salehi 2010+*)

$\mathrm{I}\Delta_0 \nvdash \mathcal{HC}\mathfrak{on}(\mathrm{I}\Delta_0)$.

Theorem (*S. Salehi 2010+*)

$(\mathrm{I}\Delta_0 + \bigwedge_j \Omega_j) \nvdash \mathcal{HC}\mathfrak{on}(\mathrm{I}\Delta_0)$.

New Results 2

Theorem (*S. Salehi 2011+*)

$\mathrm{I}\Delta_0 \nvdash \mathcal{HC}\mathfrak{on}(S)$. $\qquad S \subseteq_{\text{finite}} \mathrm{I}\Delta_0$

Corollary $\quad \mathrm{I}\Delta_0 \nvdash \mathcal{HC}\mathfrak{on}(U)$, for some $U \in \Pi_1(\mathrm{I}\Delta_0)$.

Theorem (*S. Salehi 2011+*)

$\mathrm{I}\Delta_0 + \bigwedge_j \Omega_j \nvdash \mathcal{HC}\mathfrak{on}(S)$. $\qquad S \subseteq_{\text{finite}} \mathrm{I}\Delta_0$

Corollary $\quad \mathrm{I}\Delta_0 + \bigwedge_j \Omega_j \nvdash \mathcal{HC}\mathfrak{on}(U)$, for some $U \in \Pi_1(\mathrm{I}\Delta_0)$.

Thank You!

Thanks to

The Participants . . . . For Listening...

and

The Organizers . . . For Taking Care of Everything...

SAEEDSALEHI.ir

On Herbrand Consistency of Bounded Arithmetics Logic Colloquium 2011 (16/16)