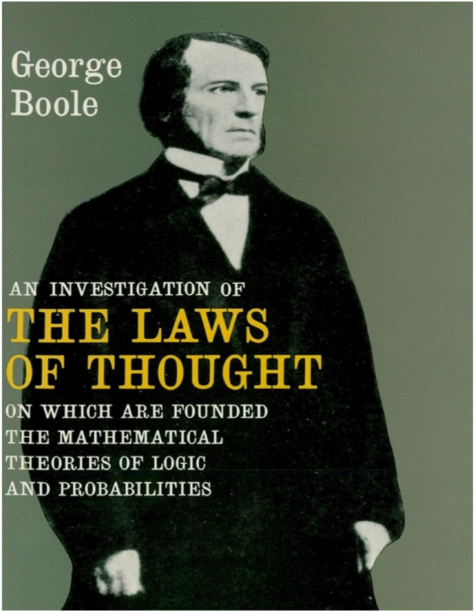
A Quick Introduction to MATHEMATICAL LOGIC

SAEED SALEHI

Frontiers Summer School in Mathematics

Boolean Algebras, 23 August 2021



AN INVESTIGATION

OF

THE LAWS OF THOUGHT

ON WHICH ARE FOUNDED

THE MATHEMATICAL THEORIES OF LOGIC
AND PROBABILITIES

BY

GEORGE BOOLE, L.L.D.

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Boolean Algebras

Associativity

$$a\wedge(b\wedge c)\equiv(a\wedge b)\wedge c, \quad a\vee(b\vee c)\equiv(a\vee b)\vee c$$

Commutativity

$$a\wedge b\equiv b\wedge a, \quad a\vee b\equiv b\vee a$$

Distributivity

$$a\wedge(b\vee c)\equiv(a\wedge b)\vee(a\wedge c), \quad a\vee(b\wedge c)\equiv(a\vee b)\wedge(a\vee c)$$

Idempotence

$$a\wedge a\equiv a, \quad a\vee a\equiv a$$

Truth and Falsum

$$a\vee(\neg a)\equiv\top, \quad a\wedge\top\equiv a, \quad a\wedge(\neg a)\equiv\bot, \quad a\vee\bot\equiv a$$

de Morgan's Laws

$$\neg(a\wedge b)\equiv(\neg a)\vee(\neg b), \quad \neg(a\vee b)\equiv(\neg a)\wedge(\neg b)$$

More on Boolean Algebras

Example

(i) It immediately follows from the axioms that
$a \equiv a \wedge \top \equiv a \wedge (p \vee \neg p) \equiv (a \wedge p) \vee (a \wedge \neg p)$.

(ii) The *absorbing properties* of truth and falsum, i.e., $a \vee \top \equiv \top$ and $a \wedge \bot \equiv \bot$ follow also from the axioms. We show the former:
$a \vee \top \equiv a \vee (a \vee \neg a) \equiv (a \vee a) \vee (\neg a) \equiv a \vee (\neg a) \equiv \top$.

(iii) One can also prove *the absorption laws*: $a \wedge (a \vee b) \equiv a$ and $a \vee (a \wedge b) \equiv a$. Let us show the latter by using (ii) above:
$a \vee (a \wedge b) \equiv (a \wedge \top) \vee (a \wedge b) \equiv a \wedge (\top \vee b) \equiv a \wedge (b \vee \top) \equiv a \wedge \top \equiv a$.

(iv) The *double negation law* $\neg\neg a \equiv a$ can be proved as follows:
$\neg\neg a \equiv (\neg\neg a) \wedge \top \equiv (\neg\neg a) \wedge (a \vee \neg a) \equiv (\neg\neg a \wedge a) \vee (\neg\neg a \wedge \neg a) \equiv$
$(\neg\neg a \wedge a) \vee (\bot) \equiv (a \wedge \neg\neg a) \vee (a \wedge \neg a) \equiv a \wedge (\neg\neg a \vee \neg a) \equiv a \wedge \top \equiv a$.

Propositional Logic by Boolean Algebras

(Classical) Logic is . . .

$$\boxed{P \to Q \quad \equiv \quad \neg P \vee Q}$$

$$(p \to p \vee q) \equiv (\neg p \vee [p \vee q]) \equiv ([p \vee \neg p] \vee q) \equiv (\top \vee q) \equiv \top$$

$$(p \wedge q \to p) \equiv (\neg [p \wedge q] \vee p) \equiv ([\neg p \vee \neg q] \vee p) \equiv (\top \vee \neg q) \equiv \top$$

$$\boxed{P \leftrightarrow Q \quad \equiv \quad (\neg P \vee Q) \wedge (P \vee \neg Q)}$$

$$\neg (p \leftrightarrow q) \equiv \neg [(\neg p \vee q) \wedge (p \vee \neg q)] \equiv \neg (\neg p \vee q) \vee \neg (p \vee \neg q) \equiv$$
$$(p \wedge \neg q) \vee (\neg p \wedge q) \equiv (p \vee q) \wedge (\neg p \vee \neg q) \equiv (\neg p \leftrightarrow q) \equiv (p \leftrightarrow \neg q)$$

A Puzzle by Boolean Algebras

P says that "*Q is lying*", and Q says that "*both P and Q tell the truth*".

Who is lying and who tells the truth?

► P says $\neg Q$ | Q says $P \wedge Q$ ◄

$P \equiv \neg Q$ | $Q \equiv P \wedge Q$

$$\begin{cases} P \equiv \neg Q \equiv \neg(P \wedge Q) \equiv \neg P \vee \neg Q \equiv \neg\neg Q \vee \neg Q \equiv \top. \\ Q \equiv P \wedge Q \equiv \neg Q \wedge Q \equiv \bot. \end{cases}$$

(P says $\neg Q$) and (Q says $P \wedge Q$) imply that

P says THE TRUTH and Q LIES!

Another Puzzle by Boolean Algebras

P says that "*either P or Q tells the truth*", and
Q says that "*P tells the truth if and only if Q does so*".
Who is lying and who tells the truth?

- P says $P \vee Q$ $\qquad$ Q says $P \leftrightarrow Q$

$P \equiv P \vee Q$ $\qquad$ $Q \equiv P \leftrightarrow Q$

$$\begin{cases} P \equiv P \vee Q \equiv P \vee (P \leftrightarrow Q) \equiv P \vee [(\neg P \vee Q) \wedge (P \vee \neg Q)] \equiv \\ P \vee \neg Q \equiv (P \vee Q) \vee \neg Q \equiv \top. \\ Q \equiv P \leftrightarrow Q \equiv (P \vee Q) \leftrightarrow Q \equiv (P \vee Q) \rightarrow Q \equiv P \rightarrow Q \ldots \end{cases}$$

(P says $P \vee Q$) and (Q says $P \leftrightarrow Q$) imply that
P says THE TRUTH and Q ??!

Logic Again

- If $\mathscr{A} \to \mathscr{B}$, and if $\mathscr{B}$, then can we infer that $\mathscr{A}$?

  $(\neg a \vee b) \wedge b \equiv b \not\to a$ x

- If $\mathscr{A} \to \mathscr{B}$, and if $\mathscr{A}$, then can we infer that $\mathscr{B}$?

  $(\neg a \vee b) \wedge a \equiv b \wedge a \to b$ ✓

- If $\mathscr{A} \to \mathscr{B}$, and if $\neg\mathscr{B}$, then can we infer that $\neg\mathscr{A}$?

  $(\neg a \vee b) \wedge \neg b \equiv \neg a \wedge \neg b \to \neg a$ ✓

- If $\mathscr{A} \to \mathscr{B}$, and if $\neg\mathscr{A}$, then can we infer that $\neg\mathscr{B}$?

  $(\neg a \vee b) \wedge \neg a \equiv \neg a \not\to \neg b$ x

https://www.wolframalpha.com/

Completeness of Boolean Algebras

Theorem (Completeness)

If $a \equiv b$ is valid according to the truth-table semantics, then it is provable from the axioms. ■

The completeness of Propositional Logic with respect to truth-table semantics follows from Completeness Theorem.

For example, the validity of the formula $[(p \to q) \to p] \to p$, Peirce's Law, can be proved by first translating $a \to b$ to $\neg a \vee b$, and then showing the equivalence $(\neg[\neg(\neg p \vee q) \vee p] \vee p) \equiv \top$ by the above axioms.

Some Exercises (1)

Three boxes are presented to you.
One contains gold, the other two are empty.
Each box has a message on its door:

Box 1 The gold is not here.

Box 2 The gold is not here.

Box 3 The gold is in Box 2.

Only one message is true; the other two are false.
Which box has the gold?

Some Exercises (2)

In Boolean Algebras define the connective

$$p \to q \equiv \neg p \vee q$$

Prove that:

- $[a \to (b \to a)] \equiv \top$
- $[a \to (b \to c)] \equiv [(a \to b) \to (a \to c)]$
- $(\neg b \to \neg a) \equiv (a \to b)$

Prove or Disprove:

- $(a \to b \wedge c) \equiv (a \to b) \wedge (a \to c)$
- $(b \vee c \to a) \equiv (b \to a) \wedge (c \to a)$
- $(a \to b \vee c) \equiv (a \to b) \vee (a \to c)$
- $(b \wedge c \to a) \equiv (b \to a) \wedge (c \to a)$

Some Exercises (3)

In Boolean Algebras define the connective

$$p \triangle q \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$$

Prove that:

- $(a \triangle a) \equiv \bot$
- $\neg(a \triangle b) \equiv (a \triangle \neg b) \equiv (\neg a \triangle b)$
- $a \triangle (b \triangle c) \equiv (a \triangle b) \triangle c$

Prove or Disprove:

- $a \wedge (b \triangle c) \equiv (a \wedge b) \triangle (a \wedge c)$
- $a \vee (b \triangle c) \equiv (a \vee b) \triangle (a \vee c)$
- $a \rightarrow (b \triangle c) \equiv (a \rightarrow b) \triangle (a \rightarrow c)$
- $a \rightarrow (b \triangle c) \equiv (a \rightarrow b \vee c) \wedge (a \rightarrow \neg b \vee \neg c)$