A Stumble of the Genius: Gödel's ω-Consistency

Saeed Salehi

𝕀ℙ𝕄, **IAL**: A Two-Day Seminar on
Mathematical Logic and Its Applications

22–23 May 2024

in the loving memory of S. Naser Hashemi (who died in 2021)

The Point of ω–Consistency.

- Craig Smoryński (1985); Springer, *Self-Reference and Modal Logic.*

  "Remark: One weakness of Gödel's original work was his introduction of the semantic notion of ω-consistency. I find this notion to be pointless, but I admit many proof theorists take it seriously." (p. 158)

pointless: maybe

semantic: never!

The Necessity of ω–Consistency.

- Kurt Gödel (1931).

  "we can … replace the assumption of ω-consistency by the following: The proposition "[T] is inconsistent" is not [T]-PROVABLE." (p. 195)

- Barkley Rosser (1936); *The Journal of Symbolic Logic* 1_3:87–91, Extensions of Some Theorems of Gödel and Church.

  "… a modification is made in Gödel's proofs of his theorems … it is proved that simple consistency implies the existence of undecidable propositions …" (p. 87)

ANOTHER INVENTION OF GÖDEL.

- S. (2020); *Reports on Mathematical Logic* 55:73–85,
  On Rudimentarity, Primitive Recursivity and Representability.

  "Primitive recursive functions are what were called 'rekursiv' by Kurt Gödel in his seminal 1931 paper … The main features of the primitive recursive functions …

  1. They are *computable* … However, we now know that they do not make up the whole … computable functions …
  2. They are *representable* in … formal arithmetical theories. It is now known that, more generally, (only) recursive functions are representable …
  3. Theories whose *set of axioms* are primitive recursive and extend a base theory (such as Robinson's Arithmetic $\boldsymbol{Q}$), are incomplete. It was later found out that this holds more generally for recursively enumerable extensions of $\boldsymbol{Q}$."

  (p. 74)

Some Stumbles of the Genius.

- Yuri Gurevich (1982); Bulletin AMS 7_1:273–277, *Review* : *The decision problem: Solvable classes of quantificational formula* **&** *Unsolvable classes of quantificational formulas.*

1. The Second Incompleteness Theorem: J. von Neumann$_{1930}$.
2. Unsolvability of Logical Decidability Problem: A. Turing$_{1937}$ & A. Church$_{1936}$. K. Gödel$_{1932}$: $\exists^*\forall^2\exists^*$decidable; $\forall^3\exists^*$reducible.
3. Independence of AC & G/CH from ZF set theory: P. Cohen$_{1963}$.
   Gödel: 1940
4. An Still Unverified Claim$_{1932}$: *every satisfiable* $\forall^2\exists^*$ [Gödel class] *sentence* ⟪with $=$⟫ *is finitely satisfiable.*

Gödel's notes on incompleteness.

In his handnotes, Gödel proved the first incompleteness theorem for *sound* theories.

- Jan von Plato (2020); *Can mathematics be proved consistent? Gödel's shorthand notes & lectures on incompleteness*, Springer.

  "The Gödel notes show stages of the development of his ideas. The clearest turning point is one connected to the Königsberg conference. Before that, Gödel's argument was to give a truth definition for propositions of *Principia Mathematica*, ... Gödel saw very clearly that the truth definition is the element in his proof that cannot be expressed within the formal system." (p. 11)

Gödel (1931).

"On formally undecidable propositions of *Principia mathematica* and related systems, I", *Collected Works I* (OUP 1986) pp. 144–195.

"The method of proof just explained can clearly be applied to any formal system that, first, ..., second, every provable formula is true in the interpretation considered." [Soundness] "The purpose of carrying out the above proof with full precision in what follows is, among other things, to replace the second of the assumptions just mentioned by a purely formal and much weaker one." (p. 151)

· · · *a controversial paragraph on the truth of Gödel Sentence(s)* · · ·

2 "We now proceed to carry out with full precision the proof sketched above."

Formal Undefinability of Truth.

- Roman Murawski (1998); *History & Phil. Logic* 19_3:153–160, Undefinability of Truth. The Problem of Priority: Tarski vs Gödel.

  "It is claimed that Tarski obtained this theorem independently though he made clear his indebtedness to Gödel's methods. On the other hand, Gödel was aware of the formal undefinability of truth in 1931, but he did not publish this result." (Abstract)

  "The theorem on the undefinability of truth was published by Alfred Tarski in his famous paper *Pojęcie prawdy w językach nauk dedukcyjnych* (1933) (German translation, 1936; English translation, 1956)."

BACK TO GÖDEL (1931).

"We now come to the goal of our discussions. Let [T] be any class of FORMULAS. ... [It] is said to be ω-consistent if there is no [formula $\varphi(x)$ with the only free variable x] such that [$T \vdash \neg\forall x\varphi(x)$ and $T \vdash \varphi(\overline{n})$ for every $n \in \mathbb{N}$]." (p. 173)

"Every ω-consistent system, of course, is consistent. As will be shown later, however, the converse does not hold."

Theorem (GÖDEL's 1st Incompleteness)

If T is a sufficiently strong theory (over a sufficiently expressive languages) then one can (algorithmically) construct a Π_1-sentence $G = \forall x\theta(x)$, with $\theta \in \Delta_0$, such that

if T is consistent, then $T \nvdash G$, *and*

if T is ω-consistent, then $T \nvdash \neg G$. ∎

Reading Gödel (1931).

> "If, instead of assuming that $[T]$ is ω-consistent, we assume only that it is consistent, then, although the existence of an undecidable proposition does not follow ⟦by the argument given above⟧, it does follow that there exists a [formula $\theta(x)$] for which it is possible neither to give a counterexample nor to prove that it holds of all numbers. For in the proof that $[\forall x\theta(x)]$ is not $[T]$-PROVABLE only the consistency of $[T]$ was used (above, page 177). Moreover … it follows … that, for every number $[n, \theta(\overline{n})]$ is $[T]$-PROVABLE and consequently that $[\neg\theta(\overline{n})]$ is not $[T]$-PROVABLE for any number $[n \in \mathbb{N}]$." (p. 179)

We have $T \vdash \theta(\overline{n})$ for every $n \in \mathbb{N}$. So, if T is consistent, then

$T \nvdash \forall x\theta(x)$ while $T \nvdash \neg\theta(\overline{n})$ for every $n \in \mathbb{N}$.

Gödel needed ω-consistency for showing $T \nvdash \neg\forall x\theta(x)$.

Still Reading Gödel (1931).

"If we adjoin [$\neg G$] to [T], we obtain a class of FORMULAS [T'] that is consistent but not ω-consistent. [T'] is consistent, since otherwise [G] would be [T]-PROVABLE. However, [T'] is not ω-consistent because [we have $T' \vdash \neg\forall x\theta(x)$, but $T' \supseteq T \vdash \theta(\overline{n})$ for every $n \in \mathbb{N}$]." (p. 179)

" [46]Of course, the existence of classes [T] that are consistent but not ω-consistent is thus proved only on the assumption that there exists some consistent [T] (that is, that [*Principia mathematica*] is consistent)." (p. 179)

$T + \neg G$ is a counterexample to many things.

This theory is Σ_1-complete, but not Σ_1-sound.

It has also *false* Gödelian (Π_1-)sentences.

The Last Page of Gödel (1931).

> "[W]e can, [in the 1st incompleteness theorem], replace the assumption of ω-consistency by the following: The proposition "[T] is inconsistent" is not [T]-PROVABLE. (Note that there are consistent [T] for which this proposition is [T]-PROVABLE.)" (p. 195)

This is because $T \vdash Con_T \longrightarrow G$ [*] (thus $T \nvdash \neg G$ if $T \nvdash \neg Con_T$) from which also the 2nd theorem ($T \nvdash Con_T$) follows.

> "The results will be stated and proved in full generality in a sequel to be published soon [...]. In that paper, also, the proof of [the 2nd incompleteness theorem], only sketched here, will be given in detail." (p. 195)

Part II never appeared; one reason being probably the "prompt acceptance of" the results.

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[*]Formalized incompleteness: if $T \vdash G \leftrightarrow \neg Prov_T(\ulcorner G \urcorner)$, then $T \vdash Con_T \longrightarrow G$.

An Interlude.

Theorem (3.4 in: Lajevardi & S. 2019, *Theoria* 85_1:8–17.)

The consistency of $T + Con_T$ ($T \nvdash \neg Con_T$) *is (also) a necessary and sufficient condition for the TRUTH (as well as the independence) of all the Gödelian* (I-am-unprovable) Π_1*-sentences.*

Proof.

If $T \vdash \neg Con_T$, then $T \vdash Prov_T(\ulcorner \bot \urcorner)$, so $T \vdash \bot \leftrightarrow \neg Prov_T(\ulcorner \bot \urcorner)$!

If $T \vdash \gamma \leftrightarrow \neg Prov_T(\bar{\gamma})$, $\gamma \in \Pi_1$, but $\mathbb{N} \vDash \neg\gamma$, then $T \vdash \neg\gamma$ (by the Σ_1-completeness), and so $T \vdash \neg Con_T$ (by $T \vdash Con_T \rightarrow \gamma$). [*] ■

- Daniel Isaacson (2011); "Necessary and Sufficient Conditions for Undecidability of the Gödel Sentence and Its Truth", in: *Logic, Mathematics, Philosophy, Vintage Enthusiasms*, Springer, 135–152.

ω-Consistency $\not\dashv\vdash$ Consistency with Con_T $\not\dashv\vdash$ Simple Consistency

Another Interlude.

- Storrs McCall (1999); *The Journal of Philosophy* 96_{10}:525–532, Can a Turing Machine Know that the Gödel Sentence is True?

  "[5] It is frequently stated that proving the nontheoremhood of $\sim G$ requires the assumption that PA is not only consistent but ω-consistent (see, for example, Mendelson, p. 143). The argument given in the text shows, however, that the weaker assumption of ordinary consistency suffices." (p. 529)

- Storrs McCall (2014); Oxford University Press, *The Consistency of Arithmetic, and other essays.*

  - A new proof is given of the consistency of arithmetic, contradicting Gödel's well-known undecidability result of 1931. http://b2n.ir/s48300

The Question Remains.

Question:
Why is ω-Consistency *purely formal and much weaker* than Soundness?

We know that "ω-consistency" is FORMALLY DEFINABLE:
$\omega\text{-}Con_T \equiv \neg\exists\chi(v)\colon Prov_T(\ulcorner\neg\forall v\chi(v)\urcorner) \wedge \forall w\, Prov_T(\ulcorner\chi(\bar{w}/v)\urcorner)$
syntactic notions (variables, terms, numerals, formulas, proofs) are definable
For Gödel (1931): $Con_T \equiv \exists x\colon Formula(x) \wedge \neg Prov_T(x)$.

While "soundness" is *not* FORMALLY DEFINABLE!

Soundness implies ω-Consistency. But NOT the other way around!

Could Gödel have meant: "to replace the second of the assumptions just mentioned by a purely formal and (thus) much weaker one"?

Some Semantic Aspects of ω–Consistency.

How Much Soundness Does (ω-)Consistency Have/Preserve6?

- Consistency $\Longrightarrow$ Π_1-Soundness (by Σ_1-completeness)
- ω-Consistency $\Longrightarrow$ Π_3-Soundness

Isaacson (2011, Theorem 17) noting that $\Sigma_2\text{-Sound}_T \equiv \Pi_3\text{-Sound}_T$.

- Con_T & $\sigma \in \Sigma_1\text{-Th}_{\mathbb{N}} \Longrightarrow Con_{T+\sigma}$ (by Σ_1-completeness)
- $\omega\text{-}Con_T$ & $\boldsymbol{\sigma} \in \Sigma_3\text{-Th}_{\mathbb{N}} \Longrightarrow \omega\text{-}Con_{T+\boldsymbol{\sigma}}$

In Isaacson (2011, Theorem 22) this is attributed to George Kreisel (2005) for $\boldsymbol{\sigma} \in \Pi_1$. Works for $\boldsymbol{\sigma} \in \Pi_2$ and $\boldsymbol{\sigma} \in \Sigma_3$ too.

Weakness of ω–Consistency.

- Leon Henkin (1954); *The Journal of Symbolic Logic* 19_3:183–196, A Generalization of the Concept of ω-Consistency
- Georg Kreisel, Mathematical Reviews (MR63324) 1955.

Attributed to Kreisel (1955) in Isaacson (2011, Proposition 19):

Proof.

Let ω-Con$_T$, and put $K \in \Sigma_3$ satisfy $\mathbf{PA} \vdash K \leftrightarrow \neg\omega\text{-}Con_T(\ulcorner K \urcorner)$.

Note that $\omega\text{-}Con_T(x) \in \Pi_3$.[†]

If $\mathbb{N} \vDash K$, then $\mathbb{N} \vDash \neg\omega\text{-}Con_{T+K}$, contrary to what was shown above!

So, $\mathbb{N} \nvDash K$, and $T+K$ is an ω-consistent but (Σ_3-)*un*sound theory! ■

The classic proof for Undefinability of Truth; just put $\omega\text{-}Con_T(x)$ in the place of $Tr(x)$. Kreisel's K is the Liar's sentence for it.

[†] $\omega\text{-}Con_T(x) \equiv \forall \chi(v) [\![\neg Prov_{T+x}(\ulcorner \neg\forall v \chi(v) \urcorner) \vee \exists w \neg Prov_{T+x}(\ulcorner \chi(\bar{w}/v) \urcorner)]\!]$

More Semantic Aspects of ω–Consistency.

How Much Soundness Does (ω-)Consistency Have Exactly?

- Consistency $\Longrightarrow$ Π_1-Soundness
  Consistency $\not\Longleftarrow$ Σ_1-Soundness

$U = T + \neg G$

- ω-Consistency $\Longrightarrow$ Π_3-Soundness
  ω-Consistency $\not\Longleftarrow$ Σ_3-Soundness

$U = T + K$

- Soundness $\Longrightarrow$ ω-Consistency
  Σ_m-Soundness $\not\Longrightarrow$ ω-Consistency, $\forall m$

Yet More Semantic Aspects of ω–Consistency.

How Much Truth Does (ω-)Consistency Preserve Exactly?

- Con_T & $\sigma \in \Sigma_1\text{-Th}_{\mathbb{N}} \Longrightarrow Con_{T+\sigma}$ (by Σ_1-completeness)
  Con_U & $\pi \in \Pi_1\text{-Th}_{\mathbb{N}} \not\Longrightarrow Con_{U+\pi}$
  Put $U = T + \neg Con_T$, and $\pi = Con_U$. Then $\mathbb{N} \models \pi$ and $U \vdash \neg\pi$.

- $\omega\text{-}Con_T$ & $\boldsymbol{\sigma} \in \Sigma_3\text{-Th}_{\mathbb{N}} \Longrightarrow \omega\text{-}Con_{T+\boldsymbol{\sigma}}$ ($\& \neg\, \omega\text{-}Con_{T+\neg\boldsymbol{\sigma}}$)
  $\omega\text{-}Con_U$ & $\boldsymbol{\pi} \in \Pi_3\text{-Th}_{\mathbb{N}} \not\Longrightarrow Con_{U+\boldsymbol{\pi}}$[‡]
  Put $U = T + K$, and $\pi = \neg K$. Then $\pi \in \Pi_3\text{-Th}_{\mathbb{N}}$ and $U \vdash \neg\pi$.

[‡]This is not a misprint: adding a true Π_3-sentence to an ω-consistent theory may not even yield a consistent theory!

SOME SYNTACTIC ASPECTS OF ω–CONSISTENCY.

- $Con_T \Longrightarrow \forall\psi\colon Con_{T+\psi} \vee Con_{T+\neg\psi}$ LINDENBAUM's Lemma
- $\omega\text{-}Con_T \Longrightarrow \forall\psi\colon \omega\text{-}Con_{T+\psi} \vee\ \omega\text{-}Con_{T+\neg\psi}$

  ISAACSON (2011, Theorem 21)
- $Con_T \wedge Complete_T \not\Longrightarrow T{=}\mathrm{Th}_{\mathbb{N}}$

  Put $\mathcal{M} \vDash S{+}\neg Con_S$ with $\mathbb{N} \vDash Con_S$, and $T{=}\mathrm{Th}_{\mathcal{M}}$.
- $\omega\text{-}Con_T \wedge Complete_T \Longrightarrow T{=}\mathrm{Th}_{\mathbb{N}}$

  ISAACSON (2011, Theorem 20)

Corollaries:

$\mathrm{Lim}_{\subseteq}\,\mathrm{Con} = Con$, but $\mathrm{Lim}_{\subseteq}\,\omega\text{-}\mathrm{Con} \neq \omega\text{-}Con$.

$Con_T \Longleftrightarrow \exists M(M \models T)$, but $\nexists\,\mathcal{C}\colon \omega\text{-}Con_T \Longleftrightarrow \exists M \in \mathcal{C}(M \models T)$.

More Syntactic Aspects of ω–Consistency.

- $Con_T \Longrightarrow Con_T(\text{“}\neg Con_T\text{”})$ Gödel's 2nd Theorem
- $\omega\text{-}Con_T \Longrightarrow \omega\text{-}Con_T(\text{“}\neg\omega\text{-}Con_T\text{”})$ $\mathbb{G}_2(\omega\text{-}Con)$

  George S. Boolos (1993, page xxxi) [*The Logic of Provability*, Cambridge University Press] says that this follows from

  John B. Rosser (1937) [Gödel Theorems for Non-Constructive Logics, *Journal of Symbolic Logic* 2_3:129–137].

- If $T \vdash \gamma \leftrightarrow \neg Prov_T(\ulcorner\gamma\urcorner)$, then $T \vdash \gamma \leftrightarrow Con_T$.
- If $T \vdash \zeta \leftrightarrow \neg Con_T(\ulcorner\zeta\urcorner)$, then $T \vdash \zeta \leftrightarrow \neg Con_T$.
- If $T \vdash \kappa \leftrightarrow \neg\omega\text{-}Con_T(\ulcorner\kappa\urcorner)$, then $T \vdash \kappa \leftrightarrow \neg\omega\text{-}Con_T$.

Yet More Syntactic Aspects of ω–Consistency.

- $Con_T \Longrightarrow \exists\mathfrak{R}\text{:}\ Con_{T+\mathfrak{R}} \wedge Con_{T+\neg\mathfrak{R}}$

Rosser (1936, Theorem II)

10 May 2022, MoPA weekly seminars:

⊛ $\omega\text{-}Con_T \Longrightarrow \exists\rho\text{:}\ \omega\text{-}Con_{T+\rho} \wedge \omega\text{-}Con_{T+\neg\rho}$??

If $\mathbb{N} \vDash T$, then $\rho = K$ works, since $\mathbb{N} \vDash T+\neg K$, so $\omega\text{-}Con_{T+\neg K}$.

If $T+\omega\text{-}Con_T$ is ω-consistent, then $\rho = \omega\text{-}Con_T$ by $\mathbb{G}_2(\omega\text{-}Con)$.

If $T+\omega\text{-}Con_T$ is not ω-consistent, then … ?!

✓ Rosser's Theorem for ω-Con.

- S. (2024?); *History and Philosophy of Logic*, On Gödel's "Much Weaker" Assumption.

Thank You!

Thanks to

The Participants .... For Listening · · ·

and

The Organizers, For Taking Care of Everything · · ·