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Congruence preserving functions of Wilke's tree algebras

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Abstract. As a framework for characterizing families of regular languages of binary trees, Wilke introduced a formalism for defining binary trees that uses six many-sorted operations involving letters, trees and contexts. In this paper a completeness property of these operations is studied. It is shown that all functions involving letters, binary trees and binary contexts which preserve congruence relations of the free tree algebra over an alphabet, are generated by Wilke's functions, if the alphabet contains at least seven letters. That is to say, the free tree algebra over an alphabet with at least seven letters is affine-complete. The proof yields also a version of the theorem for ordinary one-sorted term algebras: congruence preserving functions on contexts and members of a term algebra are substitution functions, provided that the signature consists of constant and binary function symbols only, and contains at least seven symbols of each rank. Moreover, term algebras over signatures with at least seven constant symbols are affine-complete.

1. Introduction

A new framework for characterizing families of tree languages was introduced by Wilke [15] which can be regarded as a combination of the *universal algebraic* framework of Steinby [11, 12] and Almeida [1], in the case of binary trees, which is based on *syntactic algebras* and of the *syntactic monoid/semigroup* framework of Thomas [14] and Nivat and Podelski [7, 8]. It is based on three-sorted algebras whose signature Σ consists of six operation symbols involving the sorts Alphabet, Tree and Context. Binary trees over an alphabet are represented by terms over Σ, namely as Σ-terms of sort Tree. A *tree algebra* is a Σ-algebra satisfying every identity that consists of two Σ-terms representing the same tree or context. Wilke [15] axiomatized these algebras by four identities. The *syntactic tree algebra congruence relation* of a tree language is defined in a natural way (Definition 2.1 below.) The Tree-sort component of the *syntactic tree algebra* of a tree language is the syntactic algebra of the language in the sense of [12], while its Context-component is the syntactic semigroup of the tree language, cf. [14]. A rather comprehensive study of tree algebras and Wilke's formalism has been initiated by Steinby and Salehi [10].

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In this paper we give a detailed proof of what was claimed, without presenting the full proof, in Theorem 1 of [9]: Wilke's functions generate all congruence preserving operations on the term algebra of trees, when the alphabet contains at least seven letters. A one-sorted version of this theorem, presented in Section 3 below, is interesting by itself: every congruence preserving function on contexts and members of a term algebra is a substitution function, when the signature consists of constant and binary function symbols and contains at least seven symbols of each rank.

2. Preliminaries

For an alphabet A, let Σ^{A} be the signature which contains a constant symbol c_a and a binary function symbol f_a for every $a \in A$, that is $\Sigma^{\mathrm{A}} = (\Sigma^{\mathrm{A}})_0 \cup (\Sigma^{\mathrm{A}})_2$, where $(\Sigma^{\mathrm{A}})_0 = \{c_a \mid a \in A\}$ and $(\Sigma^{\mathrm{A}})_2 = \{f_a \mid a \in A\}$. The set of *binary trees* over A, denoted by T_{A}, is defined inductively by:

- $c_a \in T_{\mathrm{A}}$ for every $a \in A$;
- $f_a(t_1, t_2) \in T_{\mathrm{A}}$ whenever $t_1, t_2 \in T_{\mathrm{A}}$ and $a \in A$.

A binary tree language over an alphabet A is any subset of T_{A}.

Fix a new symbol ξ which does not appear in A. *Binary contexts* over A are binary trees over $A \cup \{\xi\}$ in which ξ appears exactly once as a leaf. The set of non-unit binary contexts over A, denoted by C_{A}, can be defined inductively by:

- $f_a(t, \xi), f_a(\xi, t) \in C_{\mathrm{A}}$ whenever $a \in A$, $t \in T_{\mathrm{A}}$, and
- $f_a(t, p), f_a(p, t) \in C_{\mathrm{A}}$ whenever $a \in A$, $t \in T_{\mathrm{A}}$, and $p \in C_{\mathrm{A}}$.

The set of A-contexts is $C^1_{\mathrm{A}} = C_{\mathrm{A}} \cup \{\xi\}$. For contexts $p, q \in C_{\mathrm{A}}$ and tree $t \in T_{\mathrm{A}}$, the context $p(q) \in C_{\mathrm{A}}$ and tree $p(t) \in T_{\mathrm{A}}$ are obtained from p by replacing the occurrence of ξ with q and with t, respectively.

Definition 2.1. ([15], page 92) For a tree language $L \subseteq T_{\mathrm{A}}$ we define the *syntactic tree algebra congruence* relation of L, denoted by $(\approx^L_{\mathrm{A}}, \approx^L_{\mathrm{C}}, \approx^L_{\mathrm{T}})$, as follows:

(1) For any $a, b \in A$, $a \approx^L_{\mathrm{A}} b \;\equiv\; \forall p \in C^1_{\mathrm{A}}\{p(c_a) \in L \leftrightarrow p(c_b) \in L\}$ &
$\forall p \in C^1_{\mathrm{A}} \forall t_1, t_2 \in T_{\mathrm{A}}\{p(f_a(t_1, t_2)) \in L \leftrightarrow p(f_b(t_1, t_2)) \in L\}$.
(2) For any $p, q \in C_{\mathrm{A}}$, $p \approx^L_{\mathrm{C}} q \;\equiv\; \forall r \in C^1_{\mathrm{A}} \forall t \in T_{\mathrm{A}}\{r(p(t)) \in L \leftrightarrow r(q(t)) \in L\}$.
(3) For any $t, s \in T_{\mathrm{A}}$, $t \approx^L_{\mathrm{T}} s \;\equiv\; \forall p \in C^1_{\mathrm{A}}\{p(t) \in L \leftrightarrow p(s) \in L\}$.

Definition 2.2. ([15], page 88) *Wilke's functions* over an alphabet A are:

$$\iota^{\mathrm{A}} \colon A \to T_{\mathrm{A}} \qquad \iota^{\mathrm{A}}(a) = c_a$$
$$\kappa^{\mathrm{A}} \colon A \times T^2_{\mathrm{A}} \to T_{\mathrm{A}} \qquad \kappa^{\mathrm{A}}(a, t_1, t_2) = f_a(t_1, t_2)$$
$$\lambda^{\mathrm{A}} \colon A \times T_{\mathrm{A}} \to C_{\mathrm{A}} \qquad \lambda^{\mathrm{A}}(a, t) = f_a(\xi, t)$$

$$\rho^{\mathrm{A}}\colon A \times T_{\mathrm{A}} \to C_{\mathrm{A}} \qquad \rho^{\mathrm{A}}(a,t) = f_a(t,\xi)$$
$$\sigma^{\mathrm{A}}\colon C_{\mathrm{A}}^2 \to C_{\mathrm{A}} \qquad \sigma^{\mathrm{A}}(p_1,p_2) = p_1(p_2)$$
$$\eta^{\mathrm{A}}\colon C_{\mathrm{A}} \times T_{\mathrm{A}} \to T_{\mathrm{A}} \qquad \eta^{\mathrm{A}}(p,t) = p(t)$$

The above definition is the interpretation of the signature $\Sigma = \{\iota, \kappa, \lambda, \rho, \eta, \sigma\}$ in the 3-sorted Σ-structure $\mathbf{F}= (A, C_{\mathrm{A}}, T_{\mathrm{A}}, \Sigma)$, defined in [15] page 89.

Definition 2.3. ([9], Definition 4) A function $F\colon A^n \times C_{\mathrm{A}}^k \times T_{\mathrm{A}}^m \to X$ where $X \in \{A, C_{\mathrm{A}}, T_{\mathrm{A}}\}$ is called *congruence preserving*, if for every tree language $L \subseteq T_{\mathrm{A}}$ and for all $a_1, b_1, \ldots, a_n, b_n \in A$, $p_1, q_1, \ldots, p_k, q_k \in C_{\mathrm{A}}$, $t_1, s_1, \ldots, t_m, s_m \in T_{\mathrm{A}}$,

$$\text{if } a_1 \approx_{\mathrm{A}}^L b_1, \ldots, a_n \approx_{\mathrm{A}}^L b_n,\ p_1 \approx_{\mathrm{C}}^L q_1, \ldots, p_k \approx_{\mathrm{C}}^L q_k, \text{ and } t_1 \approx_{\mathrm{T}}^L s_1, \ldots, t_m \approx_{\mathrm{T}}^L s_m,$$

$$\text{then } F(a_1, \ldots, a_n, p_1, \ldots, p_k, t_1, \ldots, t_m) \approx_x^L F(b_1, \ldots, b_n, q_1, \ldots, q_k, s_1, \ldots, s_m),$$

where x is A, C, or T, if $X = A$, $X = C_{\mathrm{A}}$, or $X = T_{\mathrm{A}}$, respectively.

Remark 2.4. In universal algebra, the functions which preserve congruence relations of an algebra are called *congruence preserving* functions. On the other hand it is known that every congruence relation over an algebra is the intersection of some syntactic congruence relations (see Remark 2.12 of [1] or Lemma 6.2 of [12].) So, a function preserves all congruence relations of an algebra iff it preserves the syntactic congruence relations of all subsets of the algebra. This justifies the notion of congruence preserving function in our Definition 2.3, even though we require that the function preserves only the syntactic tree algebra congruence relations of tree languages, which is the case if and only if the function preserves all the congruence relations of the 3-sorted Σ-structure $\mathbf{F}$.

Definition 2.5. For sets $B_1, \ldots, B_n$, the *projection functions* $\pi_j^n\colon B_1 \times \cdots \times B_n \to B_j$ are defined by $\pi_j^n(b_1, \ldots, b_n) = b_j$. Each element $b \in B_j$ determines the *constant function* $B_1 \times \cdots \times B_n \to B_j$ defined by $(b_1, \ldots, b_n) \mapsto b$.

Let $\mathcal{B}$ be a collection of sets, and let C be a collection of functions of the form $B_1 \times \cdots \times B_n \to B$ for any $B_1, \ldots, B_n, B \in \mathcal{B}$. The *Pclone* generated by C is the smallest class of functions of the form $B_1 \times \cdots \times B_n \to B$, for some $B_1, \ldots, B_n, B \in \mathcal{B}$, denoted by $\mathrm{Pclone}\langle C\rangle$, that contains C and the projection and constant functions and is closed under the composition of functions. cf. the definition of clone in [6].

It is easy to see that all functions in the Pclone generated by Wilke's functions are congruence preserving.

The main result of the present paper is ([9], Theorem 1): For an alphabet A which contains at least seven letters, every congruence preserving function over A is in the Pclone generated by Wilke's functions.

More precisely, we prove the following theorems in Section 4.

Theorem 2.6. *If $|A| \geq 3$, then for all $n, m, k \in \mathbb{N}$ every congruence preserving function $A^n \times C_{\mathrm{A}}^k \times T_{\mathrm{A}}^m \to A$ is in $Pclone\langle\emptyset\rangle$, i.e., it is either a constant function or a projection function over A.*

Theorem 2.7. *If $|A| \geq 7$, then for all $n, m, k \in \mathbb{N}$ every congruence preserving functions $A^n \times C_{\mathrm{A}}^k \times T_{\mathrm{A}}^m \to T_{\mathrm{A}}$ is in $Pclone\langle\{\iota^A, \kappa^A, \eta^A\}\rangle$.*

Theorem 2.8. *If $|A| \geq 7$, then for all $n, m, k \in \mathbb{N}$ every congruence preserving function $A^n \times C_{\mathrm{A}}^k \times T_{\mathrm{A}}^m \to C_{\mathrm{A}}$ is in $Pclone\langle\{\iota^A, \kappa^A, \eta^A, \lambda^A, \rho^A, \sigma^A\}\rangle$.*

Remark 2.9. An algebra is called *congruence-primal* or *hemi-primal*, if all its congruence preserving functions are term functions, and is called *affine-complete*, if all its congruence preserving functions are polynomials, see e.g. [6]. Our main theorems imply that if $|A| \geq 7$, then the 3-sorted tree algebra $\mathbf{F}= (A, C_{\mathrm{A}}, T_{\mathrm{A}}, \Sigma)$ is affine-complete. Moreover, Theorem 2 of [9] states that any term algebra whose signature contains at least 7 constant symbols is affine-complete. We note that since in term algebras polynomials coincide with term functions, a term algebra is affine-complete iff it is congruence-primal.

3. Congruence preserving functions on contexts

In this section, Theorem 2 of [9] is generalized for contexts. For one-sorted term algebras we show that the congruence preserving functions on terms and contexts are substitution functions, when the signature consists of constant and binary function symbols and contains at least seven symbol of each rank (Theorem 3.6 below).

Our notation, as in [9], follows mainly [2, 4, 5, 6, 12, 13]. A *ranked alphabet* is a finite nonempty set of symbols each of which has a unique non-negative *arity* (or *rank*). For each $m \geq 0$, the set of m-ary symbols in a ranked alphabet Σ is denoted by Σ_m. For a set of variables X, the set of ΣX-terms, denoted by $T(\Sigma, X)$, is defined inductively by

- $\Sigma_0 \cup X \subseteq T(\Sigma, X)$, and
- $f(t_1, \ldots, t_m) \in T(\Sigma, X)$, for $f \in \Sigma_m$ $(m > 0)$ and $t_1, \ldots, t_m \in T(\Sigma, X)$.

For empty X it is simply written as T_Σ. We note that the structure $\mathcal{T}(\Sigma, X) = (T(\Sigma, X), \Sigma)$ is a Σ-algebra under the interpretation

- $c^{\mathcal{T}(\Sigma,X)} = c$, for every $c \in \Sigma_0$, and
- $f^{\mathcal{T}(\Sigma,X)}(t_1, \ldots, t_m) = f(t_1, \ldots, t_m)$, for $f \in \Sigma_m$ and $t_1, \ldots, t_m \in T(\Sigma, X)$;

is a Σ-algebra, and (T_Σ, Σ) is called the *term algebra* over Σ. Members of $T(\Sigma, X)$ are called $\Sigma X-$trees as well. That is to say, in this framework a *tree* is a *term* over a ranked alphabet and a (possibly empty) set of variables.

Fix ξ to be a new symbol which does not appear in Σ or X. A ΣX-context is a $\Sigma(X \cup \{\xi\})$-term in which ξ appears exactly once. The set of ΣX-contexts is denoted by $C^1(\Sigma, X)$, and $C(\Sigma, X) = C^1(\Sigma, X) \setminus \{\xi\}$ is the set of non-unit ΣX-contexts. Again for empty X we write C_Σ and C^1_Σ for $C(\Sigma, \emptyset)$ and $C^1(\Sigma, \emptyset)$, respectively.

If $p, q \in C^1_\Sigma$ and $t \in T_\Sigma$, then $p(q) \in C^1_\Sigma$ and $p(t) \in T_\Sigma$ are obtained from p by replacing the occurrence of ξ with q and with t, respectively. By convention $p(\xi) = p$.

For $L \subseteq T_\Sigma$, let $\approx^L$ be the syntactic congruence relation of L ([11, 12]), i.e., the greatest congruence on the term algebra T_Σ saturating L. For $t, t' \in T_\Sigma$, the relation $t \approx^L t'$ holds when $(p(t) \in L \iff p(t') \in L)$ for every $p \in C^1_\Sigma$. Another syntactic congruence of the language L, denoted by $\sim^L$, is defined on C_Σ: for $p, q \in C_\Sigma$, $p \sim^L q$ if $(r(p(t)) \in L \iff r(q(t)) \in L)$ for every $r \in C^1_\Sigma$ and $t \in T_\Sigma$, cf. [13, 14].

The following lemma is an immediate consequence of the above definitions.

Lemma 3.1. *For $L \subseteq T_\Sigma$ and $p, q \in C_\Sigma$, $p \sim^L q$ iff $p(t) \approx^L q(t)$ for every $t \in T_\Sigma$.*

In [9], congruence preserving functions of the form $(T_\Sigma)^n \to T_\Sigma$ were defined. Here we extend the definition to functions involving contexts as well:

Definition 3.2. Functions $F\colon (C_\Sigma)^m \times (T_\Sigma)^n \to T_\Sigma$ and $F'\colon (C_\Sigma)^m \times (T_\Sigma)^n \to C_\Sigma$ are called *congruence preserving*, if for all contexts $p_1, q_1, \ldots, p_m, q_m \in C_\Sigma$, trees $t_1, s_1, \ldots, t_n, s_n \in T_\Sigma$, and subsets $L \subseteq T_\Sigma$, whenever $p_1 \sim^L q_1, \ldots, p_m \sim^L q_m, t_1 \approx^L s_1, \ldots, t_n \approx^L s_n$, then

$$F(p_1, \ldots, p_m, t_1, \ldots, t_n) \approx^L F(q_1, \ldots, q_m, s_1, \ldots, s_n)$$

and

$$F'(p_1, \ldots, p_m, t_1, \ldots, t_n) \sim^L F'(q_1, \ldots, q_m, s_1, \ldots, s_n).$$

Let $\{\varrho_1, \varrho_2, \varrho_3, \ldots\}$ be a set of unary function symbols disjoint from Σ, and $\Sigma\{\varrho_1, \ldots, \varrho_m\}$ be the signature Σ augmented by $\{\varrho_1, \ldots, \varrho_m\}$.

Definition 3.3. Let $r \in T_{\Sigma\{\varrho_1,\ldots,\varrho_m\}}$ be a term. We present r as $r[\varrho_1, \ldots, \varrho_m]$ to emphasize the appearances of ϱ_i's. For contexts $p_1, \ldots, p_m \in C_\Sigma$, the term $r[p_1, \ldots, p_m] \in T_\Sigma$ is obtained from r by replacing all the occurrences of $\varrho_i(t)$, for any $t \in T_{\Sigma\{\varrho_1,\ldots,\varrho_m\}}$, with $p_i(t)$ for all $i \in \{1, 2, \ldots, m\}$.

We call the function $(C_\Sigma)^m \to T_\Sigma$ defined by $(p_1, \ldots, p_m) \mapsto r[p_1, \ldots, p_m]$ for all $p_1, \ldots, p_m \in C_\Sigma$, a *substitution function* defined by $r[\varrho_1, \ldots, \varrho_m]$.

For a set $\{x_1, \ldots, x_n\}$ of variables, a term $t \in T\big(\Sigma\{\varrho_1, \ldots, \varrho_m\}, \{x_1, \ldots, x_n\}\big)$ is also written as $t[x_1, \ldots, x_n, \varrho_1, \ldots, \varrho_m]$. For terms $s_1, \ldots, s_n$ and contexts $p_1, \ldots, p_m$, the term $t[s_1, \ldots, s_n, p_1, \ldots, p_m]$ is obtained from t by replacing all x_i's with s_i and all ϱ_j's with p_j for all i, j. The function $(T_\Sigma)^n \times (C_\Sigma)^m \to T_\Sigma$ defined by $(s_1, \ldots, s_n, p_1, \ldots, p_m) \mapsto t[s_1, \ldots, s_n, p_1, \ldots, p_m]$ for all $s_1, \ldots, s_n \in T_\Sigma$ and $p_1, \ldots, p_m \in C_\Sigma$ is also called a *substitution function* defined by t.

Similarly, the substitution function $(T_\Sigma)^n \times (C_\Sigma)^m \to C_\Sigma$ defined by a context $q[x_1, \dots, x_n, \varrho_1, \dots, \varrho_m]$ maps $(s_1, \dots, s_n, p_1, \dots, p_m)$ to $q[s_1, \dots, s_n, p_1, \dots, p_m]$ for all $s_1, \dots, s_n \in T_\Sigma$ and $p_1, \dots, p_m \in C_\Sigma$. (See also the definition of *tree substitution operation* on page 61 of [3].)

Example 3.4. The composition function of contexts $C_\Sigma \times C_\Sigma \to C_\Sigma$ defined by $(p_1, p_2) \mapsto p_1(p_2)$ is a substitution function defined by $\varrho_1(\varrho_2(\xi)) \in C_{\Sigma\{\varrho_1,\varrho_2\}}$. Also, the evaluation function $T_\Sigma \times C_\Sigma \to T_\Sigma$, $(t,p) \mapsto p(t)$, is a substitution function defined by $\varrho_1(x_1) \in T(\Sigma\{\varrho_1\}, \{x_1\})$.

The following is a classical lemma in universal algebra.

Lemma 3.5. *All substitution functions are congruence preserving.*

The rest of this section is devoted to the proof of the following Theorem:

Theorem 3.6. *Let $\Sigma = \Sigma_0 \cup \Sigma_2$ be a ranked alphabet such that $|\Sigma_0|, |\Sigma_2| \geq 7$.*

(1) *Every congruence preserving function $F\colon (T_\Sigma)^n \times (C_\Sigma)^m \to T_\Sigma$ is a substitution function, i.e., there is a term $t[x_1, \dots, x_n, \varrho_1, \dots, \varrho_m]$ in the set $T(\Sigma\{\varrho_1, \dots, \varrho_m\}, \{x_1, \dots, x_n\})$ such that for all $s_1, \dots, s_n \in T_\Sigma$ and for all $p_1, \dots, p_m \in C_\Sigma$, $F(s_1, \dots, s_n, p_1, \dots, p_m) = t[s_1, \dots, s_n, p_1, \dots, p_m]$.*

(2) *Every congruence preserving function $F\colon (T_\Sigma)^n \times (C_\Sigma)^m \to C_\Sigma$ is a substitution function, i.e., there is a context $q[x_1, \dots, x_n, \varrho_1, \dots, \varrho_m]$ in the set $C(\Sigma\{\varrho_1, \dots, \varrho_m\}, \{x_1, \dots, x_n\})$ such that for all $s_1, \dots, s_n \in T_\Sigma$ and for all $p_1, \dots, p_m \in C_\Sigma$, $F(s_1, \dots, s_n, p_1, \dots, p_m) = q[s_1, \dots, s_n, p_1, \dots, p_m]$.*

Remark 3.7. In [9], it was shown by an example that when $\Sigma = \Sigma_0 \cup \Sigma_1$ with $|\Sigma_0| = |\Sigma_1| = 1$, there is a congruence preserving function $T_\Sigma \to T_\Sigma$ which is not a substitution function. So, some lower bound must be set on $|\Sigma_0|$ in Theorem 3.6, although it is not yet known whether the bound 7 is the best possible. Here we show that the theorem does not hold for $\Sigma = \Sigma_0 \cup \Sigma_1$, with $|\Sigma_1| = 1$. For such a Σ suppose $\Sigma_1 = \{\alpha\}$ (note that no condition is set on $|\Sigma_0|$). So, $C_\Sigma = \{\alpha^n(\xi) \mid n \in \mathbb{N}\}$, and $T_{\Sigma\{\varrho_1\}} = \{\alpha^{n_1}\varrho^{m_1} \cdots \alpha^{n_k}\varrho^{m_k}(c) \mid n_1, m_1, \dots, n_k, m_k \in \mathbb{N},\, c \in \Sigma_0\}$. Hence, all the substitution functions $C_\Sigma \to T_\Sigma$ are of the form $\alpha^m(\xi) \mapsto \alpha^{\mathbf{k}m+\mathbf{n}}(\mathbf{c})$ for some fixed $\mathbf{k}, \mathbf{n} \in \mathbb{N}$ and $\mathbf{c} \in \Sigma_0$. Let, for a fixed $c_0 \in \Sigma_0$, $F\colon C_\Sigma \to T_\Sigma$ be defined by $F(\alpha^m(\xi)) = \alpha^{m^2}(c_0)$ for all $m \in \mathbb{N}$. Obviously F is not a substitution function, however we show that it is congruence preserving: for any subset $L \subseteq T_\Sigma$ and $m, n \in \mathbb{N}$, if $\alpha^m(\xi) \sim^L \alpha^n(\xi)$, then by induction on j it can be shown that $\alpha^{j+m}(c_0) \approx^L \alpha^{j+n}(c_0)$. By putting $j = m$ and once again $j = n$, we can conclude that $\alpha^{2m}(c_0) \approx^L \alpha^{2n}(c_0)$. From this and $\alpha^m(\xi) \sim^L \alpha^n(\xi)$ we infer that $\alpha^m(\alpha^{2m}(c_0)) \approx^L \alpha^n(\alpha^{2n}(c_0))$, and so on. By induction on j, it can be shown that $\alpha^{jm}(c_0) \approx^l \alpha^{jn}(c_0)$. Again by putting $j = m$ and once again $j = n$, we can infer that $\alpha^{m^2}(c_0) \approx^L \alpha^{n^2}(c_0)$, or in other words, $F(\alpha^m(\xi)) \approx^L F(\alpha^n(\xi))$.

A conference paper ([9]) was devoted to the proof of Theorem 3.6 for the functions of the form $(T_\Sigma)^n \to T_\Sigma$. The next subsection contains a detailed proof of the theorem for the functions of the form $(C_\Sigma)^n \to T_\Sigma$. In the last subsection we give a proof of the theorem in its claimed generality.

3.1. Congruence preserving functions $(C_\Sigma)^n \to T_\Sigma$**.** In this rather technical subsection, we provide the necessary definitions and lemmas for proving Theorem 3.18 below, which are generalizations of Definition 6 through Theorem 2 of [9].

Definition 3.8. A function $\delta\colon \{\varrho_1, \dots, \varrho_m\} \to C_\Sigma$ is called a *C-interpretation*. The extension $\delta^*\colon T_{\Sigma\{\varrho_1,\dots,\varrho_m\}} \to T_\Sigma$ of such a C-interpretation is defined by

- $\delta^*(c) = c$ for $c \in \Sigma_0$,
- $\delta^*(\varrho_i(t)) = \delta(\varrho_i)(\delta^*(t))$ for $t \in T_{\Sigma\{\varrho_1,\varrho_2,\dots,\varrho_m\}}$, and
- $\delta^*(f(t_1, \dots, t_n)) = f(\delta^*(t_1), \dots, \delta^*(t_n))$ for $f \in \Sigma_n$, $t_1, \dots, t_n \in T_{\Sigma\{\varrho_1,\varrho_2,\dots,\varrho_m\}}$.

In other words $\delta^*(t) = t[\delta(\varrho_1), \dots, \delta(\varrho_m)]$ for any $t[\varrho_1, \dots, \varrho_m] \in T_{\Sigma\{\varrho_1,\dots,\varrho_m\}}$.

A function $F\colon C_\Sigma \to T_{\Sigma\{\varrho_1,\dots,\varrho_m\}}$ is said to be *congruence preserving*, if for every C-interpretation δ, $\delta^* \circ F\colon C_\Sigma \to T_\Sigma$ is congruence preserving.

The notion of *subtree* is the same as of *subterm* in Universal Algebra.

Definition 3.9. Let p and q be non-unit contexts, and t be a term.

(1) p is a *subcontext* of t, if $p(s)$ is a subtree of t for some tree s.
(2) p is a *subcontext* of q, if either p is a subtree of q or $p(s)$ is a subtree of q for some tree s.
(3) q is *independent* from p if for every context r and every tree or context s, if q is a subcontext of $r(p(s))$, then q is a subcontext of either r or s.
(4) q is *non-overlapping*, if for every context r and tree or context s such that q is not a subcontext of r or s, q occurs only once as a subcontext of $r(q(s))$.
(5) q is *independent* from t, if for every context r, if q is a subcontext of $r(t)$, then q is a subcontext of r.
(6) t is *independent* from q, if for every context r and every tree or context s, if t is a subtree of $r(q(s))$, then t is a subtree of either r or s.

Example 3.10. Suppose $f \in \Sigma_2$, and $a, b \in \Sigma_0$.

(1) $q = f(f(a, f(\xi, a)), a)$ is not independent from $p = f(b, f(f(a, \xi), a))$, since q is a subcontext of $p(f(a, a)) = f(b, f(f(a, f(a, a)), a))$, and that is because $q(a) = f(f(a, f(a, a)), a)$ is a subtree of $p(f(a, a))$.
(2) $f(a, f(a, f(\xi, a)))$ and $f(b, f(b, f(\xi, b)))$ are non-overlapping and independent from each other.
(3) $q = f(f(\xi, a), a)$ is not non-overlapping, since $f(\xi, a)(q) = f(f(f(\xi, a), a), a)$ has two q subcontexts.

Lemma 3.11. *For $p, q \in C_\Sigma$ and $t \in T_\Sigma$, q is independent from p iff p is independent from q, and q is independent from t iff t is independent from q.*

Proof. (1): Assume q is independent from p and p is a subcontext of $r(q(s))$ for a context r and a term or context s such that p is not a subcontext of r or s. We note that p can not be a subcontext of q, since otherwise there would have been a subcontext u of q, and a tree or context v such that $u(p(v)) = q$, and hence by the independence of q from p, q should have been a subcontext of either u or v, a contradiction. Hence by the above assumptions we can infer the existence of a subcontext of q, call it u, and a context v such that either $u(p) = q(v)$ or $p(u) = v(q)$. Both of these possibilities lead to contradictions since they imply that q must be a subcontext of u. Hence, independence is a symmetric relation on contexts.

(2): Assume q is independent from t and t is a subtree of $r(q(s))$ for contexts r, s such that t is not a subtree of r or s. We note that q can not be a subcontext of t by the independence of q from t. Hence, there must exist a subcontext u of q and a term s' such that $u(t) = q(s')$. Then by the independence of q from t, q must be a subcontext of u as well, a contradiction.

(3): Assume t is independent from q and q is a subcontext of $u(t)$ for a context u such that q is not a subcontext of u. Then either q is a subcontext of t or t is a subtree of q. Apparently, t can not be a subtree of q because by the assumption t is independent from q. Also, q can not be a subcontext of t since from the existence of a subcontext u of t and a subtree s of t such that $t = u(q(s))$ and from the independence of t from q we must have that t is a subtree of either u or s, both of which lead to contradiction. □

Being independent from a set of trees or contexts, means being independent from each member of the set.

Proposition 3.12. *Let Σ and Σ' be ranked alphabets such that $\Sigma' = \Sigma'_0 \cup \Sigma'_2$, $\Sigma \subseteq \Sigma'$, and $|\Sigma_2|, |\Sigma_0| \geq m$ for some $m \geq 1$. Then for any $D \subset C_{\Sigma'} \cup T_{\Sigma'}$ such that $|D| < m$, there exist a non-overlapping context in C_Σ and a term in T_Σ which are independent from D.*

Proof. For every $f \in \Sigma_2$, and $c \in \Sigma_0$, define $\mathbf{p}_n^{f,c}$ by induction on n:

$$\mathbf{p}_1^{f,c} = f(c, \xi),\ \mathbf{p}_{n+1}^{f,c} = f(\mathbf{p}_n^{f,c}, c),\ \text{and let } \mathbf{t}_n^{f,c} = \mathbf{p}_n^{f,c}(c).$$

Obviously every $\mathbf{p}_n^{f,c}$ is non-overlapping. We show that there are $n \in \mathbb{N}$ and $f \in \Sigma_2$, $c \in \Sigma_0$ such that $\mathbf{p}_n^{f,c}$ and $\mathbf{t}_n^{f,c}$ are independent from D:

Take n to be a natural number greater than the height of all the members of D. Take a $f \in \Sigma_2$ that does not appear as the root symbol of any member of D, the assumption $|\Sigma_2| > |D|$ enables us to pick such a symbol. For a tree t, denote the leftmost leaf of t by $\mathrm{lf}(t)$. For a context q in $C_{\Sigma'}$, we note that there is a unique subtree of q in the form $g(t_1, t_2)$ where $g \in (\Sigma')_2$ and one of the t_i's is ξ. Let $\mathrm{lf}(q)$ be $\mathrm{lf}(t_1)$ if $t_1 \neq \xi$, and $\mathrm{lf}(q) = \mathrm{lf}(t_2)$ otherwise. By $|\Sigma_0| > |D|$, there is a $c \in \Sigma_0$ which is not equal to $\mathrm{lf}(u)$ for any $u \in D$.

Assume for some context $q \in D$, a context r and a tree or context s, that $\mathbf{p}_n^{f,c}$ is a subcontext of $r(q(s))$, but not of r or s. Since the height of $\mathbf{p}_n^{f,c}$ is greater than the height of q, then either the root of q must appear in $\mathbf{p}_n^{f,c}$ or $\mathrm{lf}(q)$ must be a subtree of $\mathbf{p}_n^{f,c}$, and both of these are in contradiction with the choice of f and c. A very similar argument shows that $\mathbf{p}_n^{f,c}$ is also independent from all trees in D. This also implies that $\mathbf{t}_n^{f,c} = \mathbf{p}_n^{f,c}(c)$ is independent from D. □

For contexts u and v, the rewriting rule $u(x) \to v(x)$ when applied to a term t, changes some subtree $u(t')$ of t, for a term t', to $v(t')$. Recall that (cf. [5]) $\Delta^*_{\{u(x)\to v(x)\}}(t)$, for a term t, is the set of descendants of t under the rewriting rule $u(x) \to v(x)$.

Lemma 3.13. *Let $F\colon C_\Sigma \to T_\Sigma$ be congruence preserving. If for $u, v \in C_\Sigma$, v is non-overlapping and independent from $\{u, F(u)\}$, then $F(v) \in \Delta^*_{\{u(x)\to v(x)\}}(F(u))$. Moreover, $F(v)$ results from $F(u)$ by replacing some subcontexts u with v.*

Proof. Denote the closure of $\{F(u)\}$ under the rewriting rule $u(x) \to v(x)$ by L, i.e., $L = \Delta^*_{\{u(x)\to v(x)\}}(F(u))$. Since v is non-overlapping and independent from $\{u, F(u)\}$, no application of the rule $u(x) \to v(x)$ results in a new subcontext of the form u, and all the v's appearing in the members of L (as subcontexts) are obtained by applying the rewriting rule $u(x) \to v(x)$. So $u \approx^L v$, and then $F(u) \approx^L F(v)$ which implies that $F(v) \in L$ since $F(u) \in L$. The second statement is straightforward. □

In what follows, we suppose $\Sigma = \Sigma_0 \cup \Sigma_2$ and $|\Sigma_2|, |\Sigma_0| \geq 7$.

Lemma 3.14. *Let $F\colon C_\Sigma \to T_{\Sigma\{\varrho_1,\ldots,\varrho_k\}}$ be congruence preserving (for a $k \in \mathbb{N}$) and $u, v \in C_\Sigma$. If v is non-overlapping and independent from $\{u, F(u)\}$, then $F(v)$ results from $F(u)$ by replacing some of its subcontexts u with v.*

Proof. By Proposition 3.12, there are non-overlapping $w, w' \in C_\Sigma$ such that w is independent from $\{u, F(u), v, F(v)\}$, and w' is independent from $\{w, u, F(u), v, F(v)\}$.

Define the C-interpretation $\delta\colon \{\varrho_1, \varrho_2, \ldots, \varrho_k\} \to C_\Sigma$ by setting $\delta(\varrho_i) = w$ for all $i \in \{1, \ldots, k\}$. By the choice of w, v is independent from $\{u, \delta^*(F(u))\}$. So we can apply Lemma 3.13 to infer that $\delta^*(F(v))$ results from $\delta^*(F(u))$ by replacing some subcontexts u with v. Note that $F(v)$ is obtained by substituting all w's in $\delta^*(F(v))$ by members of $\{\varrho_1, \ldots, \varrho_k\}$. The same is true about $F(u)$ and $\delta^*(F(u))$.

The positions of $\delta^*(F(v))$ in which w appears are exactly the same positions of $\delta^*(F(u))$ in which w appears (by the choice of w). So, the positions of $F(v)$ in which a member of $\{\varrho_1, \ldots, \varrho_k\}$ appears are exactly the same positions of $F(u)$ in which a member of $\{\varrho_1, \ldots, \varrho_k\}$ appears. We claim that members of $\{\varrho_1, \ldots, \varrho_k\}$ that appear in identical positions of $F(u)$ and $F(v)$ are identical: if not, there are non-identical $i, j \in \{1, \ldots, k\}$ such that ϱ_i appears in $F(v)$ at some position and ϱ_j appears in $F(u)$ at the same position (of $F(u)$ and $F(v)$).

Define the C-interpretation $\gamma\colon \{\varrho_1, \ldots, \varrho_k\} \to C_\Sigma$ by $\gamma(\varrho_i) = w$, and $\gamma(\varrho_l) = w'$ for all $l \neq i$. Then w appears in $\gamma^*(F(v))$ at a position, call it $\mathtt{p}$, and w' appears in $\gamma^*(F(u))$ at the same position $\mathtt{p}$. On the other hand, since v is non-overlapping and independent from $\{u, \gamma^*(F(u))\}$, by Lemma 3.13, $\gamma^*(F(v))$ results from $\gamma^*(F(u))$ by replacing some subcontexts u with v. By the choice of w and w', such a replacement can not affect the occurrences of w or w', and hence the subcontexts of $\gamma^*(F(v))$ and $\gamma^*(F(u))$ at the position $\mathtt{p}$ must be identical, a contradiction. This proves the claim which implies that $F(v)$ results from $F(u)$ by replacing some subcontexts u with v. □

Lemma 3.15. *Let $F\colon C_\Sigma \to T_{\Sigma\{\varrho_1,\ldots,\varrho_k\}}$ be congruence preserving. Then for any $u, v \in C_\Sigma$, $F(v)$ results from $F(u)$ by replacing some subcontexts u with v.*

Proof. By Proposition 3.12, there exists a non-overlapping $w \in C_\Sigma$ independent from $\{u, F(u), v, F(v)\}$. By Lemma 3.14, $F(w)$ is obtained from $F(u)$ by replacing some subcontexts u with w, and also it results from $F(v)$ by replacing some subcontexts v with w. By the choice of w, all w's appearing in $F(w)$ have been obtained either by replacing u with w in $F(u)$ or by replacing v with w in $F(v)$. Since the only difference between $F(v)$ and $F(w)$ is in the positions of $F(w)$ where w appears, and the same is true for the difference between $F(u)$ and $F(w)$, then $F(v)$ can be obtained from $F(u)$ by replacing some of its subcontexts u (the same u's which have been replaced by w to get $F(w)$) with v. □

Lemma 3.16. *Every congruence preserving function $F\colon C_\Sigma \to T_{\Sigma\{\varrho_1,\ldots,\varrho_k\}}$ is a substitution function, i.e., there exists a term $t[\varrho_1, \ldots, \varrho_k, \varrho_{k+1}] \in T_{\Sigma\{\varrho_1,\ldots,\varrho_k,\varrho_{k+1}\}}$ such that $F(u) = t[\varrho_1, \ldots, \varrho_k, u]$ for all $u \in C_\Sigma$.*

Proof. Fix a $u_0 \in C_\Sigma$, and choose a non-overlapping $v \in C_\Sigma$ independent from $\{u_0, F(u_0)\}$. By Proposition 3.12 such a v exists. Then by Lemma 3.15, $F(v)$ results from $F(u_0)$ by replacing some subcontexts u_0 with v. Let $t \in T_{\Sigma\{\varrho_1,\ldots,\varrho_k,\varrho_{k+1}\}}$ result from $F(u_0)$ by putting ϱ_{k+1} exactly at the same positions in which u_0's are replaced with v's to get $F(v)$. By the independence of v from $\{u_0, F(u_0)\}$ such a t can be uniquely found. So, $F(u_0) = t[\varrho_1, \ldots, \varrho_k, u_0]$ and also $F(v) = t[\varrho_1, \ldots, \varrho_k, v]$, moreover all v's in $F(v)$ are obtained from t by substituting all ϱ_{k+1}'s by v. We show that for any $u \in C_\Sigma$, $F(u) = t[\varrho_1, \ldots, \varrho_k, u]$ holds: By Proposition 3.12, there exists a non-overlapping w which is independent from the set $\{u_0, F(u_0), v, F(v), u, F(u)\}$. By Lemma 3.15, $F(w)$ results from $F(v)$ by replacing some subcontexts v with w.

We claim that all v's are replaced with w's in $F(v)$ to get $F(w)$. If not, then v must be a subcontext of $F(w)$. By Lemma 3.15, $F(u_0)$ results from $F(w)$ by replacing some subcontexts w with u_0, and so by the choice of w, we can infer that v is a subcontext of $F(u_0)$ which is in contradiction with the choice of v. So the claim is proved and then we can write $F(w) = t[\varrho_1, \dots, \varrho_k, w]$. Moreover all w's in $F(w)$ are obtained from t by substituting ϱ_{k+1} by w. Again by Lemma 3.15, $F(u)$ results from $F(w)$ by replacing some w subcontexts with u. We can claim that all w's appearing in $F(w)$ are replaced with u to get $F(u)$. Since otherwise w would have been a subcontext of $F(u)$ which is in contradiction with the choice of w. This shows that $F(u) = t[\varrho_1, \dots, \varrho_k, u]$. □

The following example illustrates obtaining such a tree t in the above lemma.

Example 3.17. For a ranked alphabet Σ suppose $f \in \Sigma_2$ and $c \in \Sigma_0$. Define the function $F\colon C_\Sigma \to T_{\Sigma\{\varrho_1\}}$ by $F(p) = f\big(\varrho_1(p(c)), p(p(\varrho_1(c)))\big)$ for all $p \in C_\Sigma$. It can be easily seen that F is congruence preserving. Moreover, F is a substitution function defined by $t[\varrho_1, \varrho_2] = f\big(\varrho_1(\varrho_2(c)), \varrho_2(\varrho_2(\varrho_1(c)))\big) \in T_{\Sigma\{\varrho_1,\varrho_2\}}$. Indeed, $F(p) = t[\varrho_1, p]$ for all $p \in C_\Sigma$.

Theorem 3.18. *Every congruence preserving $F\colon (C_\Sigma)^n \to T_\Sigma$ ($n \in \mathbb{N}$) is a substitution function (recall that $\Sigma = \Sigma_0 \cup \Sigma_2$ and $|\Sigma_2|, |\Sigma_0| \geq 7$).*

Proof. We proceed by induction on n. For $n = 1$ the theorem is Lemma 3.16 with $k = 0$. For the induction step let $F\colon (C_\Sigma)^{n+1} \to T_\Sigma$ be a congruence preserving function. For any $u \in C_\Sigma$ define $F_u\colon (C_\Sigma)^n \to T_\Sigma$ by $F_u(u_1, \dots, u_n) = F(u_1, \dots, u_n, u)$. By the induction hypothesis every F_u is a substitution function, i.e., there is an $t_u[\varrho_1, \dots, \varrho_n]$ in $T_{\Sigma\{\varrho_1,\dots,\varrho_n\}}$ such that $F_u(u_1, \dots, u_n) = t_u[u_1, \dots, u_n]$ for all $u_1, \dots, u_n \in C_\Sigma$. Note that such a term t_u is unique for every u. The mapping $C_\Sigma \to T_{\Sigma\{\varrho_1,\dots,\varrho_n\}}$ defined by $u \mapsto t_u$ is also congruence preserving. Hence by Lemma 3.15, it is a substitution function. So there is a $t[\varrho_1, \dots, \varrho_n, \varrho_{n+1}]$ in $T_{\Sigma\{\varrho_1,\dots,\varrho_n,\varrho_{n+1}\}}$ such that $t_u = t[\varrho_1, \dots, \varrho_n, u]$, hence $F(u_1, \dots, u_n, u_{n+1}) = F_{u_{n+1}}(u_1, \dots, u_n) = t_{u_{n+1}}[u_1, \dots, u_n] = t[\varrho_1, \dots, \varrho_n, u_{n+1}][u_1, \dots, u_n]$. It follows that $F(u_1, \dots, u_n, u_{n+1}) = t[u_1, \dots, u_n, u_{n+1}]$ is a substitution function. □

3.2. Proof of Theorem 3.6. Here, we generalize Theorem 3.18 for the functions of the form $(T_\Sigma)^n \times (C_\Sigma)^m \to T_\Sigma$ or $(T_\Sigma)^n \times (C_\Sigma)^m \to C_\Sigma$ (Theorem 3.6 below.) We recall the following definition from [9]:

Definition 3.19. An *interpretation* of X in T_Σ is a function $\epsilon\colon X \to T_\Sigma$. Its unique extension to a Σ-homomorphism $T_\Sigma(X) \to T_\Sigma$ is denoted by ϵ^*.

Definition 3.20. A function $F\colon C_\Sigma \to T(\Sigma\{\varrho_1, \dots, \varrho_m\}, X)$ is *congruence preserving* if $\epsilon^* \circ F\colon C_\Sigma \to T_{\Sigma\{\varrho_1,\dots,\varrho_m\}}$ is congruence preserving for every interpretation $\epsilon\colon X \to T_\Sigma$ (recall Definition 3.8).

Lemma 3.21. *Every congruence preserving function $C_\Sigma \to T(\Sigma\{\varrho_1, \dots, \varrho_k\}, \{x\})$, where x is a variable, is a substitution function.*

Proof. Let $F\colon C_\Sigma \to T(\Sigma\{\varrho_1, \dots, \varrho_k\}, \{x\})$ be congruence preserving and take a $p_0 \in C_\Sigma$, and an $s \in T_\Sigma$ independent from $\{p_0, F(p_0)\}$, by Proposition 3.12. Let the interpretation $\epsilon\colon \{x\} \to T_\Sigma$ be defined by $\epsilon(x) = s$. By Lemma 3.16, $\epsilon^* \circ F$ is a substitution function, defined by an $r[\varrho_1, \dots, \varrho_k, \varrho_{k+1}]$ in $T_{\Sigma\{\varrho_1,\dots,\varrho_k,\varrho_{k+1}\}}$, i.e., $\epsilon^* F(u) = r[\varrho_1, \dots, \varrho_k, u]$ for all $u \in C_\Sigma$. By the choice of s, all the occurrences of s in $\epsilon^* F(p_0)$ result from ϵ (by replacing x with s) so we can write $F(p_0) = \epsilon^* F(p_0)[s \leftarrow x]$ (all s's are replaced with x). Let $t = r[s \leftarrow x]$ be the term in $T(\Sigma\{\varrho_1, \dots, \varrho_k, \varrho_{k+1}\}, \{x\})$ which results from r by replacing all subtrees s with x. Then $F(p_0) = t[x, \varrho_1, \dots, \varrho_k, p_0]$. We show that F is defined by t, i.e., $F(q) = t[x, \varrho_1, \dots, \varrho_k, q]$, for all $q \in C_\Sigma$. Let a $q \in C_\Sigma$ be given. By Proposition 3.12, there is an $s' \in T_\Sigma$ independent from $\{p_0, F(p_0), F(q_0), s\}$. Define the interpretation $\delta\colon \{x\} \to T_\Sigma$ by $\delta(x) = s'$. By Lemma 3.16, $\delta^* \circ F$ is a substitution function defined by an $r'[\varrho_1, \dots, \varrho_k, \varrho_{k+1}]$ in $T_{\Sigma\{\varrho_1,\dots,\varrho_k,\varrho_{k+1}\}}$. In particular $\delta^* F(p_0) = r'[\varrho_1, \dots, \varrho_k, , p_0]$, and $\delta^* F(q) = r'[\varrho_1, \dots, \varrho_k, q]$. Choose a non-overlapping q_0 independent from $\{r, r', s, s'\}$ by Proposition 3.12. From $r'[\varrho_1, \dots, \varrho_k, q_0] = \delta^* F(q_0) = \epsilon^* F(q_0)[s \leftarrow s'] = r[\varrho_1, \dots, \varrho_k, q_0][s \leftarrow s']$, and by the the choice of q_0, it follows that r' results from r by replacing all the subtrees s with s'. On the other hand, by the independence of s' from $\{q, F(q)\}$, $F(q) = \delta^* F(q)[s' \leftarrow x]$, so $F(q) = r'[\varrho_1, \dots, \varrho_k, q][s' \leftarrow x]$ which implies $F(q) = r[s \leftarrow x][\varrho_1, \dots, \varrho_k, q]$, hence $F(q) = t[\varrho_1, \dots, \varrho_k, q]$. □

Lemma 3.22. *For any set of variables X, every congruence preserving function $F\colon C_\Sigma \to T(\Sigma\{\varrho_1, \dots, \varrho_k\}, X)$ is a substitution function.*

Proof. Let $x \notin X$, and let $g\colon X \to \{x\}$ be the constant function which maps every member of X to x. It can be uniquely extended to a $\Sigma\{\varrho_1, \dots, \varrho_k\}$-homomorphism $g^*\colon T(\Sigma\{\varrho_1, \dots, \varrho_k\}, X) \to T(\Sigma\{\varrho_1, \dots, \varrho_k\}, \{x\})$. By Lemma 3.21, $g^* \circ F$ is a substitution function, defined by a $r[x, \varrho_1, \dots, \varrho_k, \varrho_{k+1}]$ in $T(\Sigma\{\varrho_1, \dots, \varrho_k, \varrho_{k+1}\}, \{x\})$. So, for every $u \in C_\Sigma$, $F(u)$ can be obtained from $r[x, \varrho_1, \dots, \varrho_k, u]$ by replacing x's with some appropriate members of X. For any $p, q \in C_\Sigma$, take some t, t' in $T(\Sigma\{\varrho_1, \dots, \varrho_k, \varrho_{k+1}\}, X)$ such that $F(p) = t[X, \varrho_1, \dots, \varrho_k, p]$ and $F(q) = t'[X, \varrho_1, \dots, \varrho_k, q]$. All we have to show is that $t = t'$ which immediately implies that $F(u) = t[X, \varrho_1, \dots, \varrho_k, u]$ for all $u \in C_\Sigma$. If not, there are $x_1, x_2 \in X$ such that for a position z of t and t', x_1 appears in t at position z, and x_2 appears in t' at the same position, note that the only difference of t and t' could be the appearance of the members of X. Take an $s \in T_\Sigma$ independent from $\{p, F(p), t, q, F(q)\}$, and an $s' \in T_\Sigma$ independent from $\{p, F(p), t, q, F(q), s\}$, by Proposition 3.12. Note that s and s' are independent from t' as well. Define the interpretation $\epsilon\colon X \to T_\Sigma$ by $\epsilon(x_1) = s$ and $\epsilon(y) = s'$ for all $y \in X \setminus \{x_1\}$. Then s appears at the position

$\mathbf{z}$ of $\epsilon^* F(p)$ and s' appears at the same position of $\epsilon^* F(q)$. On the other hand, we know from Lemma 3.16 that $\epsilon^* \circ F$ is a substitution function. This leads to a contradiction by the choice of s and s'. $\square$

Definition 3.23. For a $t \in T_\Sigma$, $\eta_t \colon C_\Sigma \to T_\Sigma$ is defined by $\eta_t(p) = p(t)$ for every $p \in C_\Sigma$. A function $F \colon C_\Sigma \to C(\Sigma\{\varrho_1, \ldots, \varrho_m\}, X)$ is *congruence preserving*, if $\eta_t \circ F \colon C_\Sigma \to T(\Sigma\{\varrho_1, \ldots, \varrho_m\}, X)$ is congruence preserving for every $t \in T_\Sigma$ (recall Definition 3.20).

Lemma 3.24. *For any set of variables X, every congruence preserving $F \colon C_\Sigma \to C(\Sigma\{\varrho_1, \ldots, \varrho_k\}, X)$ is a substitution function.*

Proof. Let $\imath \colon C(\Sigma\{\varrho_1, \ldots, \varrho_k\}, X) \to T(\Sigma\{\varrho_1, \ldots, \varrho_k\}, X \cup \{\xi\})$ be the inclusion function. The lemma immediately follows from Lemma 3.22 once we note that F is congruence preserving iff $\imath \circ F$ is congruence preserving. $\square$

With an argument very similar to the proofs of Lemmas 3.21, 3.22, and 3.24, the following lemma can be proved:

Lemma 3.25. *For any set of variables X, all congruence preserving functions of the form $T_\Sigma \to T(\Sigma\{\varrho_1, \ldots, \varrho_k\}, X)$, or $T_\Sigma \to C(\Sigma\{\varrho_1, \ldots, \varrho_k\}, X)$ are substitution functions.*

Finally, we can prove the main theorem of this section.

Theorem 3.6. *If $\Sigma = \Sigma_0 \cup \Sigma_2$ and $|\Sigma_0|, |\Sigma_2| \geq 7$, then every congruence preserving function $(T_\Sigma)^n \times (C_\Sigma)^m \to T_\Sigma$ or $(T_\Sigma)^n \times (C_\Sigma)^m \to C_\Sigma$ is a substitution function.*

Proof. Let $F \colon (T_\Sigma)^n \times (C_\Sigma)^m \to T_\Sigma$ be congruence preserving. For $m = 0$ the theorem follows from Theorem 2 of [9]. Suppose $m \neq 0$. For any $(p_1, \ldots, p_m) \in (C_\Sigma)^m$ define the function $F_{(p_1,\ldots,p_m)} \colon (T_\Sigma)^n \to T_\Sigma$ by $F_{(p_1,\ldots,p_m)}(t_1, \ldots, t_n) = F(t_1, \ldots, t_n, p_1, \ldots, p_m)$. By Theorem 2 of [9], $F_{(p_1,\ldots,p_m)}$ is a substitution function, i.e., there is a $t_{(p_1,\ldots,p_m)}[x_1, \ldots, x_n] \in T(\Sigma, \{x_1, \ldots, x_n\})$ such that for all $s_1, \ldots, s_n \in T_\Sigma$, $F_{(p_1,\ldots,p_m)}(s_1, \ldots, s_n) = t_{(p_1,\ldots,p_m)}[s_1, \ldots, s_n]$. Now, the function $F' \colon (C_\Sigma)^m \to T(\Sigma, \{x_1, \ldots, x_n\})$, $F'(p_1, \ldots, p_m) = t_{(p_1,\ldots,p_m)}$ is congruence preserving. By induction on m with an argument similar to the proof of Theorem 3.18 (and the proof of Theorem 2 in [9]) using Lemma 3.22, it can be shown that F' is a substitution function as well, i.e., there is a term $t[x_1, \ldots, x_n, \varrho_1, \ldots, \varrho_m]$ in the set $T(\Sigma\{\varrho_1, \ldots, \varrho_m\}, \{x_1, \ldots, x_n\})$ such that $F'(p_1, \ldots, p_m) = t[p_1, \ldots, p_m]$. So, $F(s_1, \ldots, s_n, p_1, \ldots, p_m) = F_{(p_1,\ldots,p_m)}(s_1, \ldots, s_n) = F'(p_1, \ldots, p_m)[s_1, \ldots, s_n] = t[s_1, \ldots, s_n][p_1, \ldots, p_m] = t[s_1, \ldots, s_n, p_1, \ldots, p_m]$ is a substitution function defined by $t[x_1, \ldots, x_n, \varrho_1, \ldots, \varrho_m]$.

Now let $F\colon (T_\Sigma)^n \times (C_\Sigma)^m \to C_\Sigma$ be congruence preserving. For $m = 0$, the theorem follows from Lemma 3.25. And for $m \neq 0$, the claim (that F is a substitution function) can be proved by an argument very similar to the one used in the previous case and making use of Lemma 3.24. □

4. Congruence preserving functions of tree algebras

In this final section we prove the main theorems of the paper. Note that as a direct consequence of Theorem 3.6, we have that for $|A| \geq 7$, every congruence preserving function of the form $T_{\mathrm{A}}^m \times C_{\mathrm{A}}^k \to T_{\mathrm{A}}$ or $T_{\mathrm{A}}^m \times C_{\mathrm{A}}^k \to C_{\mathrm{A}}$ is a substitution function, where $T_{\Sigma^{\mathrm{A}}} = T_{\mathrm{A}}$ and $C_{\Sigma^{\mathrm{A}}} = C_{\mathrm{A}}$.

4.1. Congruence preserving functions $A^n \times T_{\mathrm{A}}^m \times C_{\mathrm{A}}^k \to A$. First we note that the condition $|A| \geq 3$, in Theorem 2.6 can not be improved.

Remark 4.1. The Theorem does not hold for $|A| = 2$. For $A = \{a, b\}$, let $F\colon A \to A$ be defined by $F(a) = b$ and $F(b) = a$. The function F is obviously congruence preserving but is not a constant or projection function (cf. Remark 3 of [9]).

We aim at showing that every congruence preserving function $A^n \to A$ is either a constant or projection function, if $|A| \geq 3$. For $A' \subseteq A$, the subset $T_{A'} \subseteq T_{\mathrm{A}}$ is defined in a natural way.

Lemma 4.2. *Let $F\colon A \to A$ be a congruence preserving function and $a, b \in A$. If $F(a) \in \{a, b\}$, then $F(b) \in \{a, b\}$.*

Proof. Suppose $F(a)$ is a or b. Let $L = T_{\{a,b\}}$. Then $a \approx_{\mathrm{A}}^L b$, hence $F(a) \approx_{\mathrm{A}}^L F(b)$. Since $c_{F(a)} \in L$, then $c_{F(b)} \in L$. The fact that the only trees with height one in L are c_a and c_b, implies that $F(b)$ is either a or b. □

Lemma 4.3. *Let $F\colon A \to A$ be a congruence preserving function and $a \in A$. Then*
(1) *$F(F(a)) \in \{a, F(a)\}$, and*
(2) *if $F(a) = a$, then for every $b \in A$, $F(b) \in \{a, b\}$.*

Proof. Immediate from Lemma 4.2. □

Lemma 4.4. *If $|A| \geq 3$, then every congruence preserving function $F\colon A \to A$ has a fixed point, i.e., there is an $a \in A$ such that $F(a) = a$.*

Proof. Take an arbitray $b \in A$ and assume that neither b nor $F(b)$ are fixed points of F, i.e., $F(b) \neq b$, and $F(F(b)) \neq F(b)$. By Lemma 4.3 (1), $F(F(b)) \in \{b, F(b)\}$, so $F(F(b)) = b$. By $|A| \geq 3$, there is an $a \in A$ non-identical to b and $F(b)$. Since $F(b) \notin \{a, b\}$, then by Lemma 4.2, $F(a) \notin \{a, b\}$. Similarly from $F(F(b)) = b \notin \{a, F(b)\}$ and Lemma 4.2, one gets $F(a) \notin \{a, F(b)\}$. Hence $F(a) \notin \{a, b, F(b)\}$.

Now, let $L = T_{\{a,b,F(b)\}}$. Since $a \approx^L_{\mathrm{A}} F(b)$, then $F(a) \approx^L_{\mathrm{A}} F(F(b)) = b$. From $c_b \in L$ one can infer that $c_{F(a)} \in L$, which implies that $F(a) \in \{a, b, F(a)\}$, a contradiction. □

Lemma 4.5. *For $|A| \geq 3$, every congruence preserving function $F: A \to A$ is either a constant function or the identity function over A.*

Proof. By Lemma 4.4, there is an $a \in A$ such that $F(a) = a$. Take an arbitrary $b \in A$. By Lemma 4.3 (2), $F(b) \in \{a, b\}$. We distinguish two cases:

(1) $F(b) = b$. We show that F is the identity function. For every $c \in A$ (other than a or b) by using Lemma 4.3 (2) twice, we get $F(c) \in \{a, c\}$ and $F(c) \in \{b, c\}$, which implies that $F(c) = c$, or in other words, F is the identity function.

(2) $F(b) = a$. We show that F is the constant function that maps every member of A to a. For every $c \in A \setminus \{a, b\}$, by Lemma 4.3 (2), $F(c) \in \{a, c\}$. If $F(c) = c$, then again by Lemma 4.3 (2), $F(b) \in \{c, b\}$, that is in contradiction with $F(b) = a$. So, $F(c) = a$. □

By an argument very similar to the proof of Lemma 4.2, we can show the following lemma.

Lemma 4.6. *Let $F: A^{n+1} \to A$ be congruence preserving and $a, b, d_1, \ldots, d_n \in A$. If $F(a, d_1, \ldots, d_n) \in \{a, b\}$, then $F(b, d_1, \ldots, d_n) \in \{a, b\}$.*

Theorem 4.7. *For $|A| \geq 3$, every congruence preserving function $F: A^n \to A$, for every $n \in \mathbb{N}$, is either a constant function or a projection function over A.*

Proof. By induction on n. For $n = 1$ the theorem is Lemma 4.5. For the induction step $(n+1)$, suppose $F: A^{n+1} \to A$ is congruence preserving. For each $a \in A$, let $F_a: A^n \to A$ be defined by $F_a(a_1, \ldots, a_n) = F(a, a_1, \ldots, a_n)$. Since each such F_a is congruence preserving, by the induction hypothesis it is either a constant function or a projection function over A.

We show that either all F_a's ($a \in A$), are constant functions or all F_a's are projection functions over A. Assume this is not the case. So, there are $a, b \in A$ such that F_a is a constant function, say $F_a(a_1, \ldots, a_n) = d$ for a $d \in A$, and F_b is a projection function, say $F_b(a_1, \ldots, a_n) = a_i$. We distinguish two cases:

(1) $d \in \{a, b\}$, or $F_a(a_1, \ldots, a_n) \in \{a, b\}$. There is an $e \in A$ such that $a \neq e \neq b$, since $|A| \geq 3$. Since $F(a, e, \ldots, e) = F_a(e, \ldots, e) = d \in \{a, b\}$, then by Lemma 4.6, $e = F_b(e, \ldots, e) = F(b, e, \ldots, e) \in \{a, b\}$, a contradiction.

(2) $d \notin \{a, b\}$. In this case the relations $F(b, a, \ldots, a) = F_b(a, \ldots, a) = a \in \{a, b\}$, and $F(a, a, \ldots, a) = F_a(a, \ldots, a) = d \notin \{a, b\}$ are in contradiction with Lemma 4.6.

Hence, the claim is proved: either for every $a \in A$, F_a is a constant function, or for every $a \in A$, F_a is a projection function. We treat each case separately:

(1) All F_a's are projection functions. We show that they are all equal as well. If not, there are $a, b \in A$ such that $F_a(a_1, \dots, a_n) = a_i$ and $F_b(a_1, \dots, a_n) = a_j$ for all $a_1, \dots, a_n \in A$ where $i \neq j$. Choose a $d \in A$ non-identical to a and b. Let $(a_1, \dots, a_n) \in A^n$ be $a_k = d$ for $k \neq j$, and $a_j = a$. Then $F(a, a_1, \dots, a_n) = F_a(a_1, \dots, a_n) = a_i = d \notin \{a, b\}$, and $F(b, a_1, \dots, a_n) = F_b(a_1, \dots, a_n) = a_j = a \in \{a, b\}$. We get a contradiction by Lemma 4.6. So the claim is proved: all F_a's are equal, say to π_i^n, and hence F equals to π_{i+1}^{n+1}, that is $F(a_1, a_2, \dots, a_{n+1}) = F_{a_1}(a_2, \dots, a_{n+1}) = a_{i+1}$.

(2) All F_a's are constant functions. So for every $a \in A$, there is a (unique) $d_a \in A$ such that $F_a(a_1, \dots, a_n) = d_a$. Now the mapping $F' \colon A \to A$ defined by $a \mapsto d_a$ is congruence preserving as well, hence by Lemma 4.5, F' is either a constant function or the identity function over A. If F' is a constant function, then clearly F is also a constant function: $F(a_1, a_2, \dots, a_n, a_{n+1}) = F'(a_1)$. If F' is the identity function over A, then F is the projection function π_1^{n+1}, that is $F(a_1, a_2, \dots, a_{n+1}) = F'(a_1) = a_1$. □

In the following lemma we show that every congruence preserving $C_{\mathrm{A}} \to A$ is a constant function, when $|A| \geq 2$. A very similar proof can be applied for showing that every congruence preserving $T_{\mathrm{A}} \to A$, if $|A| \geq 2$, is a constant function as well. Theorem 2.6 follows from these observations.

Lemma 4.8. *If $|A| \geq 2$, then every congruence preserving $F \colon C_{\mathrm{A}} \to A$ is a constant function.*

Proof. Recall that $\xi \notin C_{\mathrm{A}}$. For every $a \in A$, define the sequence $\{p_n^a\}_n \subset C_{\mathrm{A}}$ inductively by $p_1^a = f_a(\xi, c_a)$, and $p_{n+1}^a = f_a(p_n^a, c_a)$. We note that for any distinct $a, b \in A$, p_m^a is independent from p_n^b for all m, n.

Firstly, we show that there is an $a \in A$ such that $F(p_1^a) = a$. Take an arbitrary $a \in A$. If $F(p_1^a) = b \neq a$, then for $L = \{p_1^a(c_a), p_1^b(c_a)\}$, the relation $p_1^a \approx_{\mathrm{C}}^L p_1^b$ holds, and so $F(p_1^a) \approx_{\mathrm{A}}^L F(p_1^b)$ or $b \approx_{\mathrm{A}}^L F(p_1^b)$ holds too. This implies that $F(p_1^b) = b$, since if $F(p_1^b) = d \neq b$, then by $d \approx_{\mathrm{A}}^L b$, the set L would have had more than two elements, like $f_d(c_b, c_a), f_d(c_d, c_a), f_b(c_d, c_a)$, etc., a contradiction. So, we showed that if $F(p_1^a) = b \neq a$, then $F(p_1^b) = b$.

Secondly, we note that there is an $a \in A$ such that $F(p_n^b) = a$ for every $b \in A$ and every natural n. Take the above claimed a with $F(p_1^a) = a$ and take a $n \in \mathbb{N}$ and $b \in A$ with $b \neq a$. Then for $L = \{p_1^a(c_a), p_n^b(c_a)\}$, no $x \in A \setminus \{a\}$ can satisfy $x \approx_{\mathrm{A}}^L a$, since otherwise, with an argument similar to the previous case, L would have had more than two elements. In particular, since $p_1^a \approx_{\mathrm{C}}^L p_n^b$ and hence $a = F(p_1^a) \approx_{\mathrm{A}}^L F(p_n^b)$, so $F(p_n^b) = a$. Now the same argument with $L' = \{p_n^a(c_a), p_1^b(c_a)\}$ shows that $F(p_n^a) = F(p_1^b) = a$.

Finally, we show that there is an $a \in A$ such that $F(p) = a$ for every $p \in C_{\mathrm{A}}$. Take the above a with $F(p_n^b) = a$ (for every $b \in A$ and natural n). Take an arbitrary

$p \in C_{\mathrm{A}}$ and suppose its height is m. There is a $b \in A$ such that p is independent from p^b_{2m} (cf. Proposition 3.12). So, for $L = \{p^b_{2m}(c_a), p(c_a)\}$ we have $p \approx^L p^b_{2m}$, and thus $F(p) \approx^L F(p^b_{2m}) = a$, and this implies that $F(p) = a$, since otherwise if $F(p) = d \neq a$, then $d \approx^L a$ implies that $p^b_{2m}(c_d) \in L$ which means that L has at least two elements of height $2m$ (namely $p^b_{2m}(c_d)$ and $p^b_{2m}(c_a)$), a contradiction. □

We have now completed providing the necessary tools for proving the main theorem of this subsection.

Theorem 2.6. *If $|A| \geq 3$, then every congruence preserving $A^n \times C^k_{\mathrm{A}} \times T^m_{\mathrm{A}} \to A$ is either a constant function or a projection function over A.*

Proof. Suppose $|A| \geq 3$. An argument similar to the one used in the proof of the previous lemmas shows that every congruence preserving $T_{\mathrm{A}} \to A$ is a constant function. By induction on m and k it can be shown that every congruence preserving $(T_{\mathrm{A}})^m \times (C_{\mathrm{A}})^k \to A$ is a constant function as well. Combining this with Theorem 4.7 gives a proof for Theorem 2.6. □

4.2. Congruence preserving functions $A^n \times T^m_{\mathrm{A}} \times C^k_{\mathrm{A}} \to T_{\mathrm{A}}/C_{\mathrm{A}}$**.** In what follows we take A to be an alphabet containing at least seven letters. By Theorem 3.6, every congruence preserving function $T^m_{\mathrm{A}} \times C^k_{\mathrm{A}} \to T_{\mathrm{A}}$ is a substitution function defined by a term $t[x_1, \dots, x_m, \varrho_1, \dots, \varrho_k]$ in $T(\Sigma^A\{\varrho_1, \dots, \varrho_k\}, \{x_1, \dots, x_m\})$, similarly every congruence preserving function $T^m_{\mathrm{A}} \times C^k_{\mathrm{A}} \to C_{\mathrm{A}}$ is a substitution function defined by a context $q[x_1, \dots, x_m, \varrho_1, \dots, \varrho_k]$ in $C(\Sigma^A\{\varrho_1, \dots, \varrho_k\}, \{x_1, \dots, x_m\})$.

By the techniques elaborated in subsection 4.1 this result can be generalized to show that every congruence preserving function $F\colon A^n \times T^m_{\mathrm{A}} \times C^k_{\mathrm{A}} \to T_{\mathrm{A}}$ is a substitution function. That is to say, for a fixed set of new symbols $\{z_1, z_2, \dots\}$ disjoint from $A \cup \{x_1, x_2, \dots, \varrho_1, \varrho_2, \dots\}$, there is a term

$$t[z_1, \dots, z_n, x_1, \dots, x_m, \varrho_1, \dots, \varrho_k] \in T(\Sigma^{A\cup\{z_1,\dots,z_n\}}\{\varrho_1, \dots, \varrho_k\}, \{x_1, \dots, x_m\}),$$

such that

$$F(a_1, \dots, a_n, s_1, \dots, s_m, p_1, \dots, p_k) = t[a_1, \dots, a_n, s_1, \dots, s_m, p_1, \dots, p_k],$$

for every $a_1, \dots, a_n \in A$, $s_1, \dots, s_m \in T_{\mathrm{A}}$, and $p_1, \dots, p_k \in C_{\mathrm{A}}$. Similarly every congruence preserving function $F'\colon A^n \times T^m_{\mathrm{A}} \times C^k_{\mathrm{A}} \to C_{\mathrm{A}}$ is a substitution function defined by a context

$$q[z_1, \dots, z_n, x_1, \dots, x_m, \varrho_1, \dots, \varrho_k] \in C(\Sigma^{A\cup\{z_1,\dots,z_n\}}\{\varrho_1, \dots, \varrho_k\}, \{x_1, \dots, x_m\}),$$

such that

$$F'(a_1, \dots, a_n, s_1, \dots, s_m, p_1, \dots, p_k) = q[a_1, \dots, a_n, s_1, \dots, s_m, p_1, \dots, p_k].$$

Obviously, the term $t[a_1, \dots, a_n, s_1, \dots, s_m, p_1, \dots, p_k]$ results from t by replacing all c_{z_j}'s with c_{a_j}'s, by replacing $f_{z_j}(s, r)$'s with $f_{a_j}(s, r)$'s, and by replacing $\varrho_j(r)$'s with $p_j(r)$'s and x_j's with s_j's, for every possible j and terms r, s. By similar replacements, the context $q[a_1, \dots, a_n, s_1, \dots, s_m, p_1, \dots, p_k]$ results from q.

Theorems 2.7 and 2.8, follow from the above observations.

Remark 4.9. For an alphabet A, Wilke's functions over A (Definition 2.2) are substitution functions: ι^A, κ^A, and η^A are defined by c_{z_1}, $f_{z_1}(x_1, x_2)$, and $\varrho_1(x_1)$ in $T(\Sigma^{A\cup\{z_1\}}\{\varrho_1\}, \{x_1, x_2\})$, respectively. Also λ^A, ρ^A, and σ^A are defined by $f_{z_1}(\xi, x_1)$, $f_{z_1}(x_1, \xi)$, and $\varrho_1(\varrho_2(\xi))$ in $C(\Sigma^{A\cup\{z_1\}}\{\varrho_1, \varrho_2\}, \{x_1\})$, respectively.

Recall that the alphabet A satisfies $|A| \geq 7$.

Theorem 2.7. *Every congruence preserving function* $A^n \times C_{\mathrm{A}}^k \times T_{\mathrm{A}}^m \to T_{\mathrm{A}}$*, is in* *Pclone*$\langle\{\iota^A, \kappa^A, \eta^A\}\rangle$.

Proof. We show that the substitution function defined by any term

$$t[z_1, \dots, z_n, x_1, \dots, x_m, \varrho_1, \dots, \varrho_k] \in T(\Sigma^{A\cup\{z_1,\dots,z_n\}}\{\varrho_1, \dots, \varrho_k\}, \{x_1, \dots, x_m\}),$$

is in Pclone$\langle\{\iota^A, \kappa^A, \eta^A\}\rangle$.

For such a t, let $\widehat{t}$ be the substitution function defined by t. The proof is by the induction on the complexity of t.

First we note that for $a \in A$, $i \in \{1, \dots, n\}$, and $j \in \{1, \dots, m\}$, and for all letters $a_1, \dots, a_n \in A$, trees $s_1, \dots, s_m \in T_{\mathrm{A}}$, and contexts $p_1, \dots, p_k \in C_{\mathrm{A}}$,

- $\widehat{x_j}$ is the projection function $(a_1, \dots, a_n, s_1, \dots, s_m, p_1, \dots, p_k) \mapsto s_j$,
- $\widehat{c_a}$ is the constant function $(a_1, \dots, a_n, s_1, \dots, s_m, p_1, \dots, p_k) \mapsto \iota^A(a)$, and
- $\widehat{c_{z_i}}$ is a combination of ι^A and a projection function, satisfying

$$(a_1, \dots, a_n, s_1, \dots, s_m, p_1, \dots, p_k) \mapsto \iota^A(a_i).$$

For the induction step, suppose for terms t and r the functions $\widehat{t}$ and $\widehat{r}$ are in Pclone$\langle\{\iota^A, \kappa^A, \eta^A\}\rangle$. For simplicity write $(a_1, \dots, a_n) = \mathbf{a}$, $(s_1, \dots, s_m) = \mathbf{s}$, and $(p_1, \dots, p_k) = \mathbf{p}$. Then for $a \in A$, $i \in \{1, \dots, n\}$, and $j \in \{1, \dots, k\}$,

- $\widehat{\varrho_j(t)}$ maps $(\mathbf{a}, \mathbf{s}, \mathbf{p})$ to $\eta^A\big(p_j, \widehat{t}(\mathbf{a}, \mathbf{s}, \mathbf{p})\big)$,
- $\widehat{f_a(t, r)}$ maps $(\mathbf{a}, \mathbf{s}, \mathbf{p})$ to $\kappa^A\big(a, \widehat{t}(\mathbf{a}, \mathbf{s}, \mathbf{p}), \widehat{r}(\mathbf{a}, \mathbf{s}, \mathbf{p})\big)$, and
- $\widehat{f_{z_i}(t, r)}$ maps $(\mathbf{a}, \mathbf{s}, \mathbf{p})$ to $\kappa^A\big(a_i, \widehat{t}(\mathbf{a}, \mathbf{s}, \mathbf{p}), \widehat{r}(\mathbf{a}, \mathbf{s}, \mathbf{p})\big)$.

Hence, $\widehat{\varrho_j(t)}$, $\widehat{f_a(t, r)}$, and $\widehat{f_{z_i}(t, r)}$ are in Pclone$\langle\{\iota^A, \kappa^A, \eta^A\}\rangle$ too. □

Theorem 2.8. *Every congruence preserving function* $A^n \times C_{\mathrm{A}}^k \times T_{\mathrm{A}}^m \to C_{\mathrm{A}}$*, is in* *Pclone*$\langle\{\iota^A, \kappa^A, \eta^A, \lambda^A, \rho^A, \sigma^A\}\rangle$.

Proof. Let $\mathcal{P} = \mathrm{Pclone}\langle\{\iota^A, \kappa^A, \eta^A, \lambda^A, \rho^A, \sigma^A\}\rangle$. Keeping the notation of the proof of Theorem 2.7, we show that for any context q the substitution function defined by q, denoted by $\widehat{q}$, is in $\mathcal{P}$. Note that for any term t, the function $\widehat{t}$ belongs to $\mathcal{P}$ as well. For $a \in A$, $i \in \{1, \dots, n\}$, $j \in \{1, \dots, k\}$, and term t,

- $\widehat{\varrho_j(\xi)}$ is the projection function $(a_1, \dots, a_n, s_1, \dots, s_m, p_1, \dots, p_k) \mapsto p_j$,
- $\widehat{f_a(\xi, t)}$ maps $(\mathbf{a}, \mathbf{s}, \mathbf{p})$ to $\lambda^A\big(a, \widehat{t}(\mathbf{a}, \mathbf{s}, \mathbf{p})\big)$,
- $\widehat{f_{z_i}(\xi, t)}$ maps $(\mathbf{a}, \mathbf{s}, \mathbf{p})$ to $\lambda^A\big(a_i, \widehat{t}(\mathbf{a}, \mathbf{s}, \mathbf{p})\big)$,
- $\widehat{f_a(t, \xi)}$ maps $(\mathbf{a}, \mathbf{s}, \mathbf{p})$ to $\rho^A\big(a, \widehat{t}(\mathbf{a}, \mathbf{s}, \mathbf{p})\big)$, and
- $\widehat{f_{z_i}(t, \xi)}$ maps $(\mathbf{a}, \mathbf{s}, \mathbf{p})$ to $\rho^A\big(a_i, \widehat{t}(\mathbf{a}, \mathbf{s}, \mathbf{p})\big)$.

So, for every elementary context q, $\widehat{q} \in \mathcal{P}$. For the induction step, suppose for a context p, $\widehat{p} \in \mathcal{P}$. Then for $a \in A$, $i \in \{1, \dots, n\}$, $j \in \{1, \dots, k\}$, and term t,

- $\widehat{\varrho_j(p)}$ maps $(\mathbf{a}, \mathbf{s}, \mathbf{p})$ to $\sigma^A\big(p_j, \widehat{p}(\mathbf{a}, \mathbf{s}, \mathbf{p})\big)$,
- $\widehat{f_a(p, t)}$ maps $(\mathbf{a}, \mathbf{s}, \mathbf{p})$ to $\sigma^A\Big(\lambda^A\big(a, \widehat{t}(\mathbf{a}, \mathbf{s}, \mathbf{p})\big), \widehat{p}(\mathbf{a}, \mathbf{s}, \mathbf{p})\Big)$,
- $\widehat{f_{z_i}(p, t)}$ maps $(\mathbf{a}, \mathbf{s}, \mathbf{p})$ to $\sigma^A\Big(\lambda^A\big(a_i, \widehat{t}(\mathbf{a}, \mathbf{s}, \mathbf{p})\big), \widehat{p}(\mathbf{a}, \mathbf{s}, \mathbf{p})\Big)$,
- $\widehat{f_a(t, p)}$ maps $(\mathbf{a}, \mathbf{s}, \mathbf{p})$ to $\sigma^A\Big(\rho^A\big(a, \widehat{t}(\mathbf{a}, \mathbf{s}, \mathbf{p})\big), \widehat{p}(\mathbf{a}, \mathbf{s}, \mathbf{p})\Big)$, and
- $\widehat{f_{z_i}(t, p)}$ maps $(\mathbf{a}, \mathbf{s}, \mathbf{p})$ to $\sigma^A\Big(\rho^A\big(a_i, \widehat{t}(\mathbf{a}, \mathbf{s}, \mathbf{p})\big), \widehat{p}(\mathbf{a}, \mathbf{s}, \mathbf{p})\Big)$.

Hence, $\widehat{\varrho_j(p)}$, $\widehat{f_a(p, t)}$, $\widehat{f_{z_i}(p, t)}$, $\widehat{f_a(t, p)}$, and $\widehat{f_{z_i}(t, p)}$ are in $\mathcal{P}$ too. □

We close the paper with an example (cf. Example 1 of [9]).

Example 4.10. Let $A = \{a, b\}$. The function $F\colon A \times T_{\mathrm{A}} \times C_{\mathrm{A}} \to C_{\mathrm{A}}$ defined by

$$F(a_1, t_1, p_1) = f_a\Big(f_{a_1}\big(f_b(c_a, c_a), \xi\big), p_1\big(f_b(t_1, c_{a_1})\big)\Big)$$

for $a_1 \in A$, $t_1 \in T_{\mathrm{A}}$ and $p_1 \in C_{\mathrm{A}}$, is a substitution function defined by

$$\mathsf{r} = f_a\Big(f_{z_1}\big(f_b(c_a, c_a), \xi\big), \varrho_1\big(f_b(x_1, c_{z_1})\big)\Big) \in T(\Sigma^{\mathrm{A} \cup \{z_1\}}\{\varrho_1\}, \{x_1\}).$$

That is to say $F(a_1, t_1, p_1) = \widehat{\mathsf{r}}(a_1, t_1, p_1)$.
Moreover, $F \in \mathrm{Pclone}\langle\{\iota^A, \kappa^A, \eta^A, \lambda^A, \rho^A, \sigma^A\}\rangle$, since

$$\widehat{\mathsf{r}}(a_1, t_1, p_1) = \sigma^A\Big(\lambda^A\Big(a, \eta^A\big(p_1, \kappa^A(b, t_1, \iota^A(a_1))\big)\Big), \rho^A\big(a_1, \kappa^A(b, \iota^A(a), \iota^A(a))\big)\Big).$$

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References

[1] J. Almeida, *On pseudovarieties, varieties of languages, filters of congruences, pseudoidentities and related topics*, Algebra Universalis **27** (1990), 333–350.
[2] L. Bachmair, *Canonical equational proofs*, Progress in Theoretical Computer Science, Birkhäuser, Boston Inc., Boston MA, 1991.
[3] Z. Fülöp and S. Vágvölgyi, *Minimal equational representations of recognizable tree languages*, Acta Informatica **34** (1997), 59–84.
[4] F. Gécseg and M. Steinby, *Tree languages*, in: Handbook of formal languages (G. Rozenberg and A. Salomaa eds), Vol. 3, Springer, Berlin 1997, pp. 1–68.

[5] M. Jantzen, *Confluent string rewriting*, EATCS Monographs on Theoretical Computer Science **14**, Springer-Verlag, Berlin 1988.
[6] K. Kaarli and A. F. Pixley, *Polynomial completeness in algebraic systems*, Chapman and Hall/CRC, Boca Raton 2001.
[7] M. Nivat and A. Podelski, *Tree monoids and recognizability of sets of finite trees*, in: Resolution of Equation in Algebraic Structures, Vol. 1 Academic Press, Boston 1989, 351–367.
[8] A. Podelski, *A monoid approach to tree languages*, in: Tree Automata and Languages (M. Nivat and A. Podelski eds), North-Holland, Amsterdam 1992, pp. 41–56.
[9] S. Salehi, *A completeness property of Wilke's tree algebras*, in: Proceedings of MFCS'2003 (B. Rovan and P. Vojtáš eds), Lect. Notes Comp. Sci. **2747** Springer-Verlag 2003, pp. 662–670.
[10] S. Salehi and M. Steinby, *Tree algebras and regular tree languages,* (in preparation).
[11] M. Steinby, *Syntactic algebras and varieties of recognizable sets*, in: Proc. CAAP'79, University of Lille 1979, pp. 226–240.
[12] M. Steinby, *A theory of tree language varieties*, in: Tree Automata and Languages (M. Nivat and A. Podelski eds), North-Holland, Amsterdam 1992, pp. 57–81.
[13] M. Steinby, *General varieties of tree languages*, Theoretical Computer Science **205** (1998), 1–43.
[14] W. Thomas, *Logical aspects in the study of tree languages*, Ninth Colloquium on Trees in Algebra and in Programming, Proc. CAAP'84 (Cambridge University Press 1984), pp. 31–51.
[15] T. Wilke, *An algebraic characterization of frontier testable tree languages*, Theoretical Computer Science **154**, (1996), 85–106.

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