Diagonalizing Out by Fixed–Points

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Cantor's Diagonal Argument: The Formulation

Theorem

No function $F : A \to \mathscr{P}(A)$ can be onto.

<u>PROOF</u>: Put $D_F = \{a \in A \mid a \notin F(a)\}$. Then

$$x \in D_F \longleftrightarrow x \notin F(x)$$

and so $D_F \neq F(\alpha)$ for any $\alpha \in A$: if $D_F = F(\alpha)$ then

$$\alpha \in D_F \longleftrightarrow \alpha \notin F(\alpha) \longleftrightarrow \alpha \notin D_F!$$ ❑

Cantor's 3rd Proof for the Uncountability of $\mathbb{R}$

JOHN FRANKS, Cantor's Other Proofs that $\mathbb{R}$ is Uncountable, *Mathematics Magazine* 83:4 (2010) 283–289. doi:10.4169/002557010X521822

Cantor's Diagonal Argument: Why *Diagonal*?

For $A = \{x, y, a, b, c, \cdots\}$ Put $F : A \to \mathscr{P}(A)$ as:

	x	y	a	b	c	$\cdots$	
$F(x)$	0	0	1	1	0	$\cdots$	$F(x) = \{x, y, a, b, c, \cdots\}$
$F(y)$	0	0	1	0	1	$\cdots$	$F(y) = \{x, y, a, b, c, \cdots\}$
$F(a)$	1	1	1	0	0	$\cdots$	$F(a) = \{x, y, a, b, c, \cdots\}$
$F(b)$	0	0	1	0	0	$\cdots$	$F(b) = \{x, y, a, b, c, \cdots\}$
$F(c)$	0	0	0	1	0	$\cdots$	$F(c) = \{x, y, a, b, c, \cdots\}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$

Cantor's Diagonal Argument: Why *Diagonal*?

Then Diagonalize Out:

	x	y	a	b	c	$\cdots$	
$F(x)$	$\boxed{0}$	0	1	1	0	$\cdots$	$F(x)=\{x,y,a,b,c,\cdots\}$
$F(y)$	0	$\boxed{0}$	1	0	1	$\cdots$	$F(y)=\{x,y,a,b,c,\cdots\}$
$F(a)$	1	1	$\boxed{1}$	0	0	$\cdots$	$F(a)=\{x,y,a,b,c,\cdots\}$
$F(b)$	0	0	1	$\boxed{0}$	0	$\cdots$	$F(b)=\{x,y,a,b,c,\cdots\}$
$F(c)$	0	0	0	1	$\boxed{0}$	$\cdots$	$F(c)=\{x,y,a,b,c,\cdots\}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\searrow$	1	1	0	1	1		$D_F=\{x,y,a,b,c,\cdots\}$

$$D_F \neq F(x), F(y), F(a), F(b), F(c), \cdots$$

Cantor's Diagonal Argument: Other Formulations

Every $F : A \to \mathscr{P}(A)$ Corresponds to a relation $\mathscr{R}_F \subseteq A \times A$ as

$$x\,\mathscr{R}_F\,y \iff y \in F(x)$$

Every binary relation $R \subseteq A \times A$ Corresponds to a function $\mathscr{F}_R : A \to \mathscr{P}(A)$ as

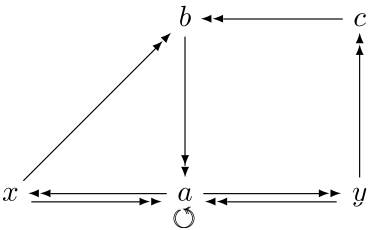
$$\mathscr{F}_R(x) = \{y \in A \mid x\,R\,y\}$$

$\mathscr{F}_{\mathscr{R}_F} = F$ and $\mathscr{R}_{\mathscr{F}_R} = R$

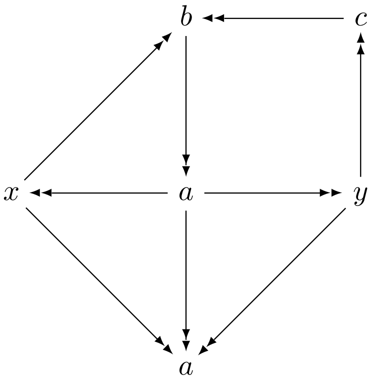
Cantor's Theorem: In Any Directed Graph There Exists a Set of Nodes Such That Is Not Equal to the Set of Outgoing Nodes of Any Fixed Node.

Cantor's Diagonal Argument: Directed Graphs

	x	y	a	b	c	
$F(x)$	0	0	1	1	0	$F(x) = \{x, y, a, b, c, \cdots\}$
$F(y)$	0	0	1	0	1	$F(y) = \{x, y, a, b, c, \cdots\}$
$F(a)$	1	1	1	0	0	$F(a) = \{x, y, a, b, c, \cdots\}$
$F(b)$	0	0	1	0	0	$F(b) = \{x, y, a, b, c, \cdots\}$
$F(c)$	0	0	0	1	0	$F(c) = \{x, y, a, b, c, \cdots\}$



Cantor's Diagonal Argument: Directed Graphs



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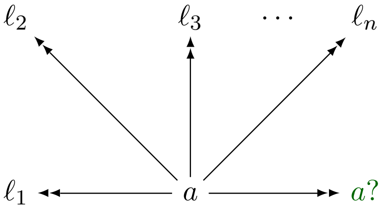
Cantor's Diagonal Argument: Directed Graphs

Cantor's Theorem: The Set of Loopless Nodes of a Directed Graph Cannot be the Set of Outgoing Nodes of Any Fixed Node.

<u>PROOF</u>: If $\{\ell_1, \ell_2, \ell_3, \cdots\} = \texttt{LoopLess} = \{x \mid aRx\}$



then $a \not R a \longleftrightarrow a \in \texttt{LoopLess} \longleftrightarrow a \in \{x \mid aRx\} \longleftrightarrow aRa!$ ❑

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Cantor's Diagonal Argument: Other Formulations

Every binary relation $R \subseteq A \times A$ Corresponds to a function

$\mathtt{C}_R : A \times A \to \{0,1\}$ as $\mathtt{C}_R(x,y) = \begin{cases} 1 & \text{if } xRy \\ 0 & \text{if } x\not{R}y \end{cases}$

Every function $f : A \times A \to \{0,1\}$ Corresponds to a binary relation $\Re_f \subseteq A \times A$ as $x\Re_f y \iff f(x,y) = 1$

$$\Re_{\mathtt{C}_R} = R \quad \text{and} \quad \mathtt{C}_{\Re_f} = f$$

Cantor's Theorem: No Function $f : A \times A \to \{0,1\}$ Can Represent All The Functions $A \to \{0,1\}$.

Cantor's Diagonal Argument: Two–Valued Functions

The function $g : A \to \{0,1\}$ is *represented* by $f : A \times A \to \{0,1\}$ (at $a \in A$) when $g(x) = f(x,a)$ holds (for every $x \in A$).

Cantor's Theorem: For Any Function $f : A \times A \to \{0,1\}$ There Is Some Function $g : A \to \{0,1\}$ Which Is Not Represented By f.

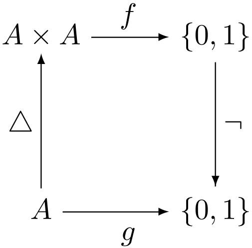
- Diagonal Function: $\triangle_A : A \to A \times A, \;\; x \mapsto \langle x, x\rangle$
- Diagonalization of $f : A \times A \to \{0,1\}$ is: $f \circ \triangle, \;\; x \mapsto f(x,x)$
- Negation Function: $\neg : \{0,1\} \to \{0,1\}, \;\; i \mapsto 1 - i$
- Anti–Diagonal Function of f is: $\neg \circ f \circ \triangle, \;\; x \mapsto \neg f(x,x)$

(Diagonalizing Out of f)

Cantor's Diagonal Argument: Two–Valued Functions

Cantor's Theorem: For Any Function $f : A \times A \to \{0,1\}$ The Anti–Diagonal Function of f Is Not Represented By f.

PROOF: For $g = \neg \circ f \circ \triangle$, $g(x) = \neg f(x,x)$



if $g(x) = f(x,\alpha)$ then $f(\alpha,\alpha) = g(\alpha) = \neg f(\alpha,\alpha)$! ❑

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Diagonal Arguments by Two–Valued Functions

▶ F. WILLIAM LAWVERE, Diagonal Arguments and Cartesian Closed Categories, *Category Theory, Homology Theory and Their Applications II*, LNM 92, Springer (1969) 134–145. Repubed in *Reprints in Theory and Applications of Categories* 15 (2006) 1–13. http://www.tac.mta.ca/tac/reprints/articles/15/tr15abs.html

▶ F. WILLIAM LAWVERE & ROBERT ROSEBRUGH, *Sets for Mathematics*, Cambridge University Press (2003).

▶ NOSON S. YANOFSKY, A Universal Approach to Self-Referential Paradoxes, Incompleteness and Fixed Points, *Bull. Symbolic Logic* 9:3 (2003) 362–386.

PARADOXES: the Liar, the strong liar, Russell, Grelling, Richard, Time Travel, and Löb;

THEOREMS: Turing, Baker–Gill–Solovay, Karnap, Gödel, Rosser, Tarski, Parikh, Kleene, Rice, and von Neumann.

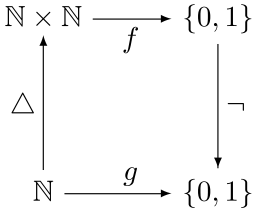
Diagonal Arguments by Two–Valued Functions

Euclid's Theorem: There Are Infinitely Many Prime Numbers.

PROOF: Let $f(n,m) =$

$$\begin{cases} 1 & \text{if all the prime factors of } (n!+1) \text{ are less than } m \\ 0 & \text{if some prime factor of } (n!+1) \text{ is greater than or equal to } m \end{cases}$$

If $\mathfrak{p} \in \mathbb{N}$ is the biggest prime then g is representable by f at $\mathfrak{p}$:

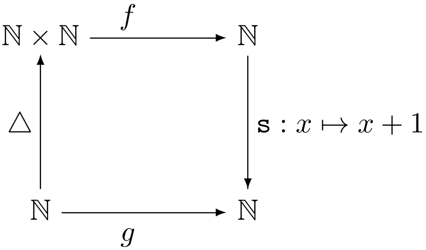


by $f(x,x)=0$ we have $g(x)=\neg f(x,x) = 1 = f(x,\mathfrak{p})$! ❑

Diagonal Arguments for Dominating Functions

Theorem: For a sequence of functions $f_0, f_1, f_2, f_3, \cdots : \mathbb{N} \to \mathbb{N}$, there exists a function $g : \mathbb{N} \to \mathbb{N}$ that dominates them all.

PROOF: Let $f(n,m) = \max_{(i \leqslant n)} f_i(m)$ and $g(x) = f(x,x) + 1$:



$x \geqslant m \Longrightarrow g(x) > \max_{(i \leqslant x)} f_i(x) \geqslant f_m(x)$. ❏

Diagonal Arguments for Dominating Functions

	0	1	2	3	4	$\cdots$	
f_0	$f_0(0)$	$f_0(1)$	$f_0(2)$	$f_0(3)$	$f_0(4)$	$\cdots$	
f_1	$f_1(0)$	$f_1(1)$	$f_1(2)$	$f_1(3)$	$f_1(4)$	$\cdots$	
f_2	$f_2(0)$	$f_2(1)$	$f_2(2)$	$f_2(3)$	$f_2(4)$	$\cdots$	
f_3	$f_3(0)$	$f_3(1)$	$f_3(2)$	$f_3(3)$	$f_3(4)$	$\cdots$	
f_4	$f_4(0)$	$f_4(1)$	$f_4(2)$	$f_4(3)$	$f_4(4)$	$\cdots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
$\searrow$	max	max	max	max	max		$+1 = g$

$$g(x) = \max\{f_i(x) \mid i \leqslant x\} + 1$$

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Variants of Diagonal Argument: Permuting Rows

For $A = \{x, y, a, b, c, \cdots\}$ and $F : A \to \mathscr{P}(A)$

	x	y	a	b	c	$\cdots$	
$F(x)$	0	0	1	1	0	$\cdots$	$F(x) = \{x, y, a, b, c, \cdots\}$
$F(y)$	0	0	1	0	1	$\cdots$	$F(y) = \{x, y, a, b, c, \cdots\}$
$F(a)$	1	1	1	0	0	$\cdots$	$F(a) = \{x, y, a, b, c, \cdots\}$
$F(b)$	0	0	1	0	0	$\cdots$	$F(b) = \{x, y, a, b, c, \cdots\}$
$F(c)$	0	0	0	1	0	$\cdots$	$F(c) = \{x, y, a, b, c, \cdots\}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$

Variants of Diagonal Argument: Permuting Rows

Permute rows by $h : A \to A$ as

$h(x) = a,\ h(y) = b,\ h(a) = y,\ h(b) = c,\ h(c) = x$

	x	y	a	b	c	$\cdots$	
$F(h(x))$	1	1	1	0	0	$\cdots$	$F(a) = \{x, y, a, b, c, \cdots\}$
$F(h(y))$	0	0	1	0	0	$\cdots$	$F(b) = \{x, y, a, b, c, \cdots\}$
$F(h(a))$	0	0	1	0	1	$\cdots$	$F(y) = \{x, y, a, b, c, \cdots\}$
$F(h(b))$	0	0	0	1	0	$\cdots$	$F(c) = \{x, y, a, b, c, \cdots\}$
$F(h(c))$	0	0	1	1	0	$\cdots$	$F(x) = \{x, y, a, b, c, \cdots\}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$

Variants of Diagonal Argument: Permuting Rows

Diagonalizing Out:

	x	y	a	b	c	$\cdots$	
$F(h(x))$	$\boxed{1}$	1	1	0	0	$\cdots$	$F(a)=\{x,y,a,b,c,\cdots\}$
$F(h(y))$	0	$\boxed{0}$	1	0	0	$\cdots$	$F(b)=\{x,y,a,b,c,\cdots\}$
$F(h(a))$	0	0	$\boxed{1}$	0	1	$\cdots$	$F(y)=\{x,y,a,b,c,\cdots\}$
$F(h(b))$	0	0	0	$\boxed{1}$	0	$\cdots$	$F(c)=\{x,y,a,b,c,\cdots\}$
$F(h(c))$	0	0	1	1	$\boxed{0}$	$\cdots$	$F(x)=\{x,y,a,b,c,\cdots\}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\searrow$	0	1	0	0	1		$D_{F\circ h}=\{x,y,a,b,c,\cdots\}$

Variants of Diagonal Argument: Permuting Rows

For any $F : A \to \mathscr{P}(A)$ and any surjection $h : A \to A$, put

$$D_{F\circ h} = \{a \in A \mid a \notin F(h(a))\}$$

If $D_{F\circ h} = F(\alpha)$ and $h(\beta) = \alpha$ (by surjectivity of h), then

$$\beta \in D_{F\circ h} \longleftrightarrow \beta \notin F(h(\beta)) \longleftrightarrow \beta \notin F(\alpha) \longleftrightarrow \beta \notin D_{F\circ h}$$

Is Any Set ($B \subseteq A$) Not In The Range Of F ($B \neq F(\square)$) In The Form Of $D_{F\circ h}$ For Some (SURJECTION) h?

ROBERT GRAY, George Cantor and Transcendental Numbers, *The American Mathematical Monthly* 101:9 (1994) 819–832.

Theorem: A real number in the interval $(0, 1)$ is transcendental if and only if it is the diagonal number of a sequence that consists of all the binary representations of algebraic reals in $(0, 1)$.

Variants of Diagonal Argument: Permuting Rows

For No Surjective h Can We Have $D_{F\circ h} = \{b, c\}$:

	x	y	a	b	c	
$F(x)$	0	0	1	1	0	$F(x) = \{x, y, a, b, c\}$
$F(y)$	0	0	1	0	1	$F(y) = \{x, y, a, b, c\}$
$F(a)$	1	1	1	0	0	$F(a) = \{x, y, a, b, c\}$
$F(b)$	0	0	1	0	0	$F(b) = \{x, y, a, b, c\}$
$F(c)$	0	0	0	1	0	$F(c) = \{x, y, a, b, c\}$

$$D_{F\circ h} = \{a \in A \mid a \notin F(h(a))\}$$

$$x \notin D_{F\circ h} \longrightarrow x \in F(h(x)) \longrightarrow h(x) = a$$

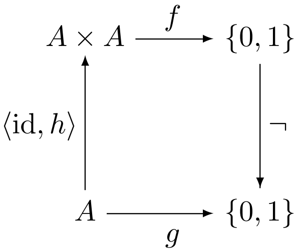
$$y \notin D_{F\circ h} \longrightarrow y \in F(h(y)) \longrightarrow h(y) = a$$

So h cannot be injective and (by A's finiteness) cannot be surjective.

Diagonalizing Out of Two–Valued Functions

Theorem: For Any Function $f : A \times A \to \{0, 1\}$ And Any Surjection $h : A \to A$ The Function $g : A \to A$, $x \mapsto \neg f(x, h(x))$ Is Not Represented By f.

PROOF:



If $g(x) = f(x, \alpha)$ and $h(\beta) = \alpha$ then

$$f(\beta, \alpha) = g(\beta) = \neg f(\beta, h(\beta)) = \neg f(\beta, \alpha)!$$ ❑

Diagonalizing Out of Two–Valued Functions

$g: A \to \{0,1\}$ by $g(x) = \neg f(x, h(x))$ is not represented by $f: A \times A \to \{0,1\}$ (i.e., $f(x,\alpha) \neq g(x)$ for any $\alpha \in A$) when h is

- surjective: $\forall\alpha\exists\beta[h(\beta) = \alpha]$

  $f(\beta,\alpha) = g(\beta) = \neg f(\beta, h(\beta)) = \neg f(\beta,\alpha)!$

- surjective w.r.t f: $\forall\alpha\exists\beta[f(x, h(\beta)) = f(x,\alpha)]$

  $f(\beta,\alpha) = g(\beta) = \neg f(\beta, h(\beta)) = \neg f(\beta,\alpha)!$

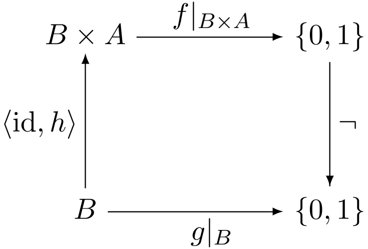
- f−surjective: $\forall\alpha\exists\beta[f(\beta, h(\beta)) = f(\beta,\alpha)]$

  $f(\beta,\alpha) = g(\beta) = \neg f(\beta, h(\beta)) = \neg f(\beta,\alpha)!$

Diagonalizing Out of Two–Valued Functions

Theorem: For $f: A\times A\rightarrow\{0,1\}$ and $B\subseteq A$ if there exists an $f-$surjective $h: B\rightarrow A$ then any $g: A\rightarrow A$ satisfying $g|_B(x)=\neg f|_{B\times A}(x,h(x))$ is not represented by f.

PROOF:



If $g(x)=f(x,\alpha)$ and $f(\beta,h(\beta))=f(\beta,\alpha)$ (by $f-$surjectivity of h) then $f(\beta,\alpha)=g(\beta)=g|_B(\beta)=\neg f|_{B\times A}(\beta,h(\beta))=\neg f(\beta,\alpha)$! ❑

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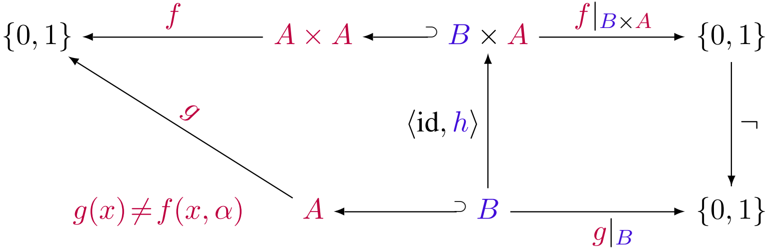
Diagonalizing Out of Two–Valued Functions

Theorem: For $f:A\times A\to\{0,1\}$, a function $g:A\to A$ is not represented by f if and and only if there exist $B\subseteq A$ and an f–surjection $h:B\to A$ such that $g|_B(x)=\neg f|_{B\times A}(x,h(x))$.



$$\forall\alpha\in A\,\exists\beta\in B\,[f(\beta,h(\beta))=f(\beta,\alpha)]$$

Thank You!

Thanks to The Participants
for Listening and for Their Patience!
and Thanks to The Organizers
For Everything!