# Rice's Theorem for First-Order R.E. Theories

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## Undecidability of the HALTING PROBLEM

It is undecidable whether

`for a Given Turing Machine and a Given Input`

The Turing Machine Ever Halts With The Input.

Language of a TM = Words with which the TM halts.

$\Updownarrow$

R.E. Language

Halting Problem $\equiv$ Membership Problem of an R.E. Language.


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## Some Other Undecidable Problems

Whether a Given R.E. Language

- has at least two distinct elements
- is empty
- has no word containing the letter $\mathfrak{b}$
- contains the fixed language $\mathfrak{L}$
- is disjoint from the fixed language $\mathfrak{L}$

Property of R.E. Languages = A Class $\mathcal{P}$ of R.E. Languages.
Non-Trivial Property $\equiv (\mathcal{P} \neq \emptyset) \& (\mathcal{P} \neq$ All R.E. Languages).


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## Non-Trivial Properties of R.E. Languages

A Set $\emptyset \subset \mathcal{Q} \subset \mathbb{N}$ such that
if $L(\texttt{TM}_n) = L(\texttt{TM}_m)$ then $n \in \mathcal{Q} \iff m \in \mathcal{Q}$.

Thus for example for $\mathcal{E} = \{k \in \mathbb{N} \mid L(\texttt{TM}_k) = \emptyset\}$ we have
either $\mathcal{E} \subseteq \mathcal{P}$ or $\mathcal{E} \cap \mathcal{P} = \emptyset$.

### Rice's Theorem

Every Non-Trivial Property of R.E. Languages is UNDECIDABLE.

The Ricean Objection: An Analogue of Rice's Theorem for First-order Theories – Igor Carboni Oliveira and Walter Carnielli (State University of Campinas, Brazil) –

Logic Journal of the IGPL (2008) 16 (6): 585-590.

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Erratum to *The Ricean Objection: An Analogue of Rice's Theorem for First-Order Theories* –

Logic Journal of the IGPL (2009) 17 (6): 803-804.

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First-Order R.E. (and FINITELY AXIOMATIZABLE) Theories...?

## Logical Properties

A First Order Axiomatizable (R.E.) Theory $T$

- Is Consistent: $T \nvdash \bot$
- Is Finitely Axiomatizable: $T \vdash\dashv \varphi$ for a sentence $\varphi$
- Implies Goldbach's Conjecture: $T \vdash \texttt{GC}$
- Is Universally Axiomatizable: $T \vdash\dashv T_\forall$
- Has a Truth Predicate: $T \vdash \Psi(\overline{\varphi}) \leftrightarrow \varphi$ for a formula $\Psi$
- Is Complete: $T \vdash \theta$ or $T \vdash \neg\theta$ for any formula $\theta$
- Is $\Sigma_1-$Complete: $T \vdash \phi$ for any TRUE $\Sigma_1-$formula $\phi$
- Has a Finite Model: $\cdots$

**Non-Trivial and Logical Properties of First-Order R.E. Theories**

A Class $\mathcal{P}$ of Theories s.t. $\emptyset \neq \mathcal{P} \neq$ All Theories, and is Logical.

$\mathrm{T}_n$ = Theory (Set of Sentences) Generated by $\mathtt{TM}_n$

[[ $\mathrm{T}_n$ = Theory (Set of Sentences) Recognized by $\mathtt{TM}_n$ ]]

**A Set $\emptyset \subset \mathcal{Q} \subset \mathbb{N}$ Such That**

If $\mathrm{T}_n \equiv \mathrm{T}_m (\mathrm{T}_n \vdash\dashv \mathrm{T}_m)$ Then $n \in \mathcal{Q} \iff m \in \mathcal{Q}$.

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**True Analogue of Rice's Theorem for First-Order R.E. Theories:**

Any Non-Trivial Logical Property of First-Order R.E. Theories is Undecidable.

A Theorem of Church: First-Order Logic is UNDECIDABLE.
Thus the CONSISTENCY of A Given Theory is UNDECIDABLE.

A Proof of Rice's Theorem for First-Order R.E. Theories
goes by
Reducing the CONSISTENCY to Non-Trivial Logical PROPERTY.

Without Loss of Generality We CAN Assume That $\mathcal{P}$ Contains NO Inconsistent Theory. (`Otherwise Take` $\mathcal{P}^{\complement}$).

Recall That $\mathcal{P}$
Either (i) Contains All Inconsistent Theory
Or (ii) Has No Inconsistent Theory.

## A Proof for Ricean Objection for First-Order R.E. Theories

Take a consistent theory $S \in \mathcal{P}$ with $S = \{S_{(0)}, S_{(1)}, S_{(2)}, \cdots\}$.
For a given Theory $T$ define the theory $f(T)$ by

$f(T)_{(k)} = S_{(k)}$ if $\neg\texttt{Proof}_T(k, \overline{\bot})$;

$f(T)_{(k)} = \bot$ if $\texttt{Proof}_T(k, \overline{\bot})$.

If $\textsc{Con}(T)$ then $f(T) = S \in \mathcal{P}$.

If $\neg\textsc{Con}(T)$ then $\neg\textsc{Con}(f(T))$, thus $f(T) \notin \mathcal{P}$.

Whence, $\textsc{Con}(T) \iff f(T) \in \mathcal{P}$.

Thus $\mathcal{P}$ cannot be Decidable, since Con is not Decidable.

QED

### Corollary

The Property of Finitely Axiomatizability of a given First-Order R.E. Theory is NOT Decidable!?

Some Non-Trivial Property of Finitely Axiomatizable Theories ($\equiv$ Sentences) is Decidable: Whether it is Derivable (Included) in a Decidable Theory (like *Presburger Arithmetic* or *Skolem Arithmetic* or *Theory of Real Closed Fields* or ...).

Thank You All for Participating and for Listening!