Mathematical Interpretations of Non-Normal Modality

SAEED SALEHI

http://saeedsalehi.info/

Moscow 2008

- Non-Normal Modal Logics
- Why Non-Normal?
- Mathematical Interpretations
- Non-Normality – Semantically

Propositional Modal Logics

Classical Propositional Calculus + Modality Axioms and Rules

Axiom:

$$(\mathtt{K}) \quad \Box(A \to B) \to (\Box A \to \Box B)$$

Rule:

$$(\mathtt{RN}) \quad \frac{A}{\Box A}$$

This base logic is denoted **K**.

Add more axioms, get stronger modal logics.

$(\mathtt{4})$ $\Box A \to \Box\Box A$; logic **K4**.

$(\mathtt{L})$ $\Box(\Box A \to A) \to \Box A$; Gödel-Löb logic **GL**.

$$(\mathtt{K}) + (\mathtt{L}) + (\mathtt{RN}) = \ \mathbf{GL} \vdash (\mathtt{4}).$$

Normal Modal Logics $\supseteq$ **K**

Modal Logics Weaker than **K**

A semantics for modal logics:

Lindenbaum-Tarski (Boolean) Algebras

$\mathcal{B} = (B, \wedge, \vee, ', \leqslant, 0, 1, \boxdot) \qquad \boxdot : B \to B$

Let T be a theory. $[\varphi]_T = \{\psi \mid T \vdash \varphi \leftrightarrow \psi\}$.

$[\varphi]_T \wedge [\psi]_T = [\varphi \wedge \psi]_T \qquad [\varphi]_T \vee [\psi]_T = [\varphi \vee \psi]_T$

$[\varphi]'_T = [\neg\varphi]_T \qquad [\varphi]_T \leqslant [\psi]_T$ iff $T \vdash \varphi \to \psi$;

$0 = [\bot]_T \qquad 1 = [\top]_T \qquad \boxdot[\varphi]_T = [\Box\varphi]_T$.

$$\text{Well–defined iff} \quad \frac{T \vdash \varphi \leftrightarrow \psi}{T \vdash \Box\varphi \leftrightarrow \Box\psi}.$$

Minimal Modal Logic **E**

CPC + Rule of Inference

$$(\mathtt{RE})\ \frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}.$$

Monotone Modal Logic **M**

CPC + Monotonicity Rule

$$(\mathtt{RM})\ \frac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$$

(or equivalently) **E** + the Axiom

$$(\mathtt{M})\ \Box(A \wedge B) \rightarrow \Box A \wedge \Box B.$$

Necessitation Modal Logic **N**

CPC + Necessitation Rule

$$(\mathtt{RN})\ \frac{\varphi}{\Box\varphi}$$

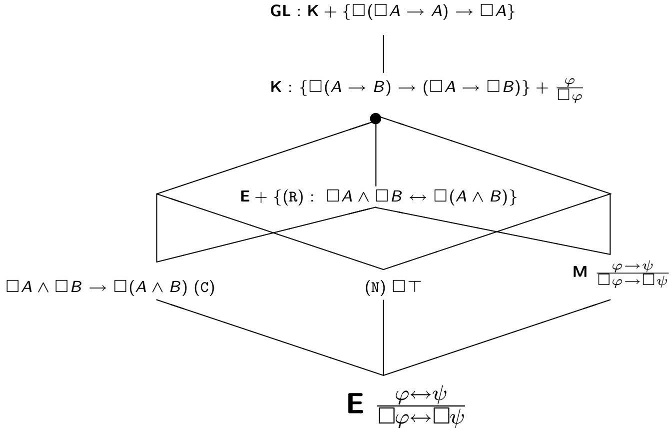
(or equivalently) **E** + the Axiom

$$(\mathtt{N})\ \Box\top.$$

Axiom $(\mathtt{C})\ \Box A \wedge \Box B \to \Box(A \wedge B)$ converse of monotonicity

---

$$\mathbf{K} = \mathbf{E} + (\mathtt{N}) + (\mathtt{M}) + (\mathtt{C}) = \mathbf{M} + \mathbf{N} + \mathtt{C}$$


$\mathbf{GL} : \mathbf{K} + \{\Box(\Box A \to A) \to \Box A\}$
$\mathbf{K} : \{\Box(A \to B) \to (\Box A \to \Box B)\} + \frac{\varphi}{\Box\varphi}$
$\mathbf{E} + \{(\mathtt{R}) : \Box A \wedge \Box B \leftrightarrow \Box(A \wedge B)\}$
$\Box A \wedge \Box B \to \Box(A \wedge B)$ $(\mathtt{C})$
$(\mathtt{N})$ $\Box\top$
$\mathbf{M}$ $\frac{\varphi \to \psi}{\Box\varphi \to \Box\psi}$
$\mathbf{E}$ $\frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Literature:

B. Chellas, *Modal Logic: An Introduction*, CUP 1990.

Philosophically ...?

No (explicit) mention in the **Handbook of Modal Logic**?

Proof–Theoretic Aspects [e.g. cut elimination] Different Systems

Let $\Box\varphi$ mean

- happening of φ with high probability
- having a strategy to force φ
- the set of consequences of φ
- cut-free provability of φ in weak arithmetics

then $\Box$ does not satisfy ($\mathtt{K}$).

High Probability

Fix a threshold $r < 1$ and let $\Box\varphi$ mean

happening of φ with probability $\geq r$.

Take an $1 \leq x < 1/\sqrt{r}$, and assume ϕ and ψ are independent with probability $x \cdot r$. Then $\Box\phi \wedge \Box\psi$.
But $\Box(\phi \wedge \psi)$ does not hold, because the probability of $\phi \wedge \psi$ is $x^2 \cdot r^2 < (1/r) \cdot r^2 = r$.
Thus $(\mathtt{C}) : \Box\phi \wedge \Box\psi \not\to \Box(\phi \wedge \psi)$ under this interpretation.

Though $(\mathtt{RE})$: $A \leftrightarrow B / \Box A \leftrightarrow \Box B$, $(\mathtt{M})$: $\Box(A \wedge B) \to \Box A \wedge \Box B$, and $(\mathtt{N})$: $\Box\top$ are valid.

Deductive Closure

For Σ a set of sentences in CPC, a Σ-valuation is a mapping $*$
$(A \wedge B)^* = A^* \cap B^*$, $(\neg A)^* = \Sigma - A^*$, and
$(\Box A)^* = \{\alpha \in \Sigma \mid A^* \vdash_{CPC} \alpha\}$.

This modal logic can be axiomatized by

- ▷ $A \rightarrow \Box A$ — reflexivity
- ▷ $\Box(A \vee \Box A) \rightarrow \Box A$ — transitivity
- ▷ $A \rightarrow B / \Box A \rightarrow \Box B$ — monotonicity

because

- ◁ $A^* \subseteq (\Box A)^*$
- ◁ $(\Box(A \vee \Box A))^* \subseteq (\Box A)^*$
- ◁ if $A^* \subseteq B^*$ then $(\Box A)^* \subseteq (\Box B)^*$

Deductive Closure

Proof of Completeness in
[P. Naumov, "On modal logic of deductive closure", *APAL* (2006)]

For $(\mathtt{C}) : \Box A \wedge \Box B \rightarrow \Box(A \wedge B)$ we should have
$(\Box A)^* \cap (\Box B)^* \subseteq (\Box(A \wedge B))^*$ which is not true:
$A^* \vdash \alpha \ \ \& \ \ B^* \vdash \alpha \not\longrightarrow \ \ A^* \cap B^* \vdash \alpha$
(put $A^* = \{\mathfrak{p}\}$, $B^* = \{\mathfrak{q}\}$, and $\alpha = \mathfrak{p} \vee \mathfrak{q}$).
Thus $\Box A \wedge \Box B \not\rightarrow \Box(A \wedge B)$.

Also $(\mathtt{N}) : \Box\top$, because $\{\alpha \in \Sigma \mid \Sigma \vdash \alpha\} = \Sigma$.

Cut-Free Provability

An example of a non-normal incompleteness:

$$\mathtt{e}:\ \frac{\varphi \leftrightarrow \Box\psi}{\Diamond\varphi \leftrightarrow \Diamond\Box\psi}; \qquad \mathtt{m}:\ \frac{\varphi \rightarrow \Box\psi}{\Diamond\varphi \rightarrow \Diamond\Box\psi};$$

$\mathtt{s}:\ \Diamond\varphi \wedge \Box\psi \rightarrow \Diamond(\varphi \wedge \Box\psi)$; $\mathtt{m}':\ \Diamond(\varphi \wedge \psi) \rightarrow \Diamond\psi$; $\mathtt{f}:\ \mathbb{G} \leftrightarrow \neg\Box\mathbb{G}$;

where $\mathbb{G}$ is a propositional constant.
Note that $\mathtt{s}$ follows from (and does not imply) $\mathtt{K4}$.

We can show a formalized second incompleteness theorem
$\vdash \Diamond\varphi \rightarrow \neg\Box\Diamond\varphi$:

Cut-Free Provability

From e, f : $\dfrac{\neg\mathbb{G} \leftrightarrow \Box\mathbb{G}}{\Diamond\neg\mathbb{G} \leftrightarrow \Diamond\Box\mathbb{G}}$, thus $\vdash \mathbb{G} \leftrightarrow \neg\Box\mathbb{G} \leftrightarrow \Diamond\neg\mathbb{G} \leftrightarrow \Diamond\Box\mathbb{G}$.

Now, $\Diamond\varphi \wedge \neg\mathbb{G} \vdash^{\mathtt{f}} \Diamond\varphi \wedge \Box\mathbb{G} \vdash^{\mathtt{s}} \Diamond(\varphi \wedge \Box\mathbb{G}) \vdash^{\mathtt{m}'} \Diamond\Box\mathbb{G} \vdash^{\uparrow} \mathbb{G}$.

So $\vdash \Diamond\varphi \rightarrow \mathbb{G}$. Then $\vdash \neg\mathbb{G} \rightarrow \Box\neg\varphi$, and by $\mathtt{m}'$: $\vdash \Diamond\neg\mathbb{G} \rightarrow \Diamond\Box\neg\varphi$.

Whence $\vdash \Diamond\varphi \rightarrow \mathbb{G} \rightarrow \Diamond\neg\mathbb{G} \rightarrow \Diamond\Box\neg\varphi \rightarrow \neg\Box\Diamond\varphi$.

By adding $\mathtt{N}$: $A/\Box A$, we can also show $\nvdash \Diamond\psi$.

Löb's Axiom – Formalized Gödel's 2nd Incompltns Thm.

$$\Diamond\psi \rightarrow \neg\Box(\psi \rightarrow \Diamond\psi)$$

$$\Diamond\psi \rightarrow \Diamond(\psi \& \neg\Diamond\psi)$$

$$\neg\Box\neg\psi \rightarrow \neg\Box(\neg\psi \vee \Diamond\psi)$$

$$\varphi = \neg\psi: \quad \neg\Box\varphi \rightarrow \neg\Box(\varphi \vee \neg\Box\varphi)$$

$$\Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi \quad !$$

By non-normal bi-modal methods we can show

$$\mathrm{I}\Delta_0 + \Omega_1 \nvdash \mathrm{HCon}(\mathrm{I}\Delta_0 + \Omega_1)$$

even stronger

$$\mathrm{I}\Delta_0 + \Omega_1 \vdash \mathrm{HCon}(\mathrm{I}\Delta_0 + \Omega_1) \rightarrow \neg \mathrm{HPr}^* \Big(\mathrm{HCon}(\mathrm{I}\Delta_0 + \Omega_1) \Big)$$

where

$\mathrm{HCon}(\mathrm{I}\Delta_0 + \Omega_1) =$ Herbrand Consistency of $\mathrm{I}\Delta_0 + \Omega_1$

$\mathrm{HPr}^*(\phi) =$ Herbrand Provability of ϕ in the cut $\mathbf{log^2}$

$\mathbf{log^2} = \{x \mid 2^{2^x} \text{ exists}\}$

Kripke (Relational) Models: $\mathcal{M} = (W, R, \vDash)$
where $R \subseteq W \times W$ and $\vDash \subseteq W \times \mathrm{Atomic\ Formulae}$; then
$w \vDash \phi$ iff $(w, \phi) \in \vDash$ for atomic ϕ
and by satisfiability conditions for more complex formulae;
$w \vDash \Box\varphi$ iff $v \vDash \varphi$ for every v with wRv.

Then `K` and `N` are valid in every Kripke model.
The Logic of Kripke Models is **K** ($\subseteq$ Normal).

Neighborhood Models: $\mathcal{M} = (W, \mathrm{N}, \mathscr{V})$
where $\mathrm{N} : W \to \mathscr{P}\mathscr{P}(W)$ - neighborhood function; and

$\mathscr{V} : \mathrm{Atomic} \to \mathscr{P}(W)$ which can be extended to all formulae:

$\mathscr{V}(\neg\phi) = W - \mathscr{V}(\phi)$; $\mathscr{V}(\phi \wedge \psi) = \mathscr{V}(\phi) \cap \mathscr{V}(\psi)$; and

$\mathscr{V}(\Box\phi) = \{w \in W \mid \mathscr{V}(\phi) \in \mathrm{N}(w)\}$.

I.O.W. $w \models \Box\phi \quad \Leftrightarrow \quad \{v \in W \mid v \models \phi\} \in \mathrm{N}(\phi)$.

Then $\mathtt{RE}$: $A \leftrightarrow B / \Box A \leftrightarrow \Box B$ is valid in every Neighborhood model.
The Logic of Neighborhood Models is **E** ($\subseteq$ Classical).

M $\langle \underline{\overline{\textit{sound\&complete}}} \rangle$ each $\mathrm{N}(w)$ closed under superset

---

N $\langle \underline{\overline{\textit{sound\&complete}}} \rangle$ each $\mathrm{N}(w) \ni W$

C $\langle \underline{\overline{\textit{sound\&complete}}} \rangle$ each $\mathrm{N}(w)$ closed under intersection

---

K $\langle \underline{\overline{\textit{sound\&complete}}} \rangle$ each $\mathrm{N}(w)$ is a filter

Neighborhood Models

There is more ...

For a Kripke Model $(W, R, \vDash)$ let $(W, \aleph, \mathscr{V})$ be defined:
$\aleph(w) = \Big\{X \subseteq W \mid X \supseteq \{v \in W \mid wRv\}\Big\}$ and
$\mathscr{V}(\phi) = \{w \in W \mid w \vDash \phi\}$.

Then each $\aleph(w)$ is a [principal] filter.

Eric Pacuit:
Neighborhood Semantics for Modal Logic
An Introduction
Course at ESSLLI 2007

❀ Thank You ! ❀

for listening ...