ω-CONSISTENCY: GÖDEL'S "MUCH WEAKER" NOTION OF SOUNDNESS

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GÖDEL (1931).

"On formally undecidable propositions of *Principia mathematica* and related systems, I", *Collected Works I* (OUP 1986) pp. 144–195.

"The method of proof just explained can clearly be applied to any formal system that, first, ..., second, every provable formula is true in the interpretation considered." [Soundness]
"The purpose of carrying out the above proof with full precision in what follows is, among other things, to replace the second of the assumptions just mentioned by a purely formal and much weaker one." (p. 151).

· · · *a controversial paragraph on the truth of Gödel Sentence(s)* · · ·

2 "We now proceed to carry out with full precision the proof sketched above."

GÖDEL'S NOTES ON INCOMPLETENESS.

In his handnotes, Gödel proved the first incompleteness theorem for *sound* theories.

- JAN VON PLATO (2020); *Can mathematics be proved consistent? Gödel's shorthand notes & lectures on incompleteness*, Springer.

"The Gödel notes show stages of the development of his ideas. The clearest turning point is one connected to the Königsberg conference. Before that, Gödel's argument was to give a truth definition for propositions of *Principia Mathematica*, ... Gödel saw very clearly that the truth definition is the element in his proof that cannot be expressed within the formal system." (p. 11).

Formal Undefinability of Truth.

- Roman Murawski (1998); Undefinability of Truth. The Problem of Priority: Tarski vs Gödel, *History and Philosophy of Logic* 19(3):153–160.

"It is claimed that Tarski obtained this theorem independently though he made clear his indebtedness to Gödel's methods. On the other hand, Gödel was aware of the formal undefinability of truth in 1931, but he did not publish this result." (Abstract).
"The theorem on the undefinability of truth was published by Alfred Tarski in his famous paper *Pojęcie prawdy w językach nauk dedukcyjnych* (1933) (German translation, 1936; English translation, 1956)."

BACK TO GÖDEL (1931).

"We now come to the goal of our discussions. Let [T] be any class of FORMULAS. ... [It] is said to be ω-consistent if there is no [formula $\varphi(x)$ with the only free variable x] such that [$T \vdash \neg\forall x \varphi(x)$ and $T \vdash \varphi(\overline{n})$ for every $n \in \mathbb{N}$]." (p. 173).

"Every ω-consistent system, of course, is consistent. As will be shown later, however, the converse does not hold."

Theorem (GÖDEL's 1st Incompleteness)

If T is a sufficiently strong theory (over a sufficiently expressive languages) then one can (algorithmically) construct a Π_1-sentence $G = \forall x \theta(x)$, with $\theta \in \Delta_0$, such that

if T is consistent, then $T \nvdash G$, *and*

if T is ω-consistent, then $T \nvdash \neg G$. ■

READING GÖDEL (1931).

"If, instead of assuming that [T] is ω-consistent, we assume only that it is consistent, then, although the existence of an undecidable proposition does not follow ⟦by the argument given above⟧, it does follow that there exists a [formula $\theta(x)$] for which it is possible neither to give a counterexample nor to prove that it holds of all numbers. For in the proof that [$\forall x\theta(x)$] is not [T]-PROVABLE only the consistency of [T] was used (above, page 177). Moreover ... it follows ... that, for every number [n, $\theta(\overline{n})$] is [T]-PROVABLE and consequently that [$\neg\theta(\overline{n})$] is not [T]-PROVABLE for any number [$n \in \mathbb{N}$]." (p. 179).

We have $T \vdash \theta(\overline{n})$ for every $n \in \mathbb{N}$. So, if T is consistent, then

$T \nvdash \forall x\theta(x)$ while $T \nvdash \neg\theta(\overline{n})$ for every $n \in \mathbb{N}$.

Gödel needed ω-consistency for showing $T \nvdash \neg\forall x\theta(x)$.

STILL READING GÖDEL (1931).

"If we adjoin [$\neg G$] to [T], we obtain a class of FORMULAS [T'] that is consistent but not ω-consistent. [T'] is consistent, since otherwise [G] would be [T]-PROVABLE. However, [T'] is not ω-consistent because [we have $T' \vdash \neg\forall x\theta(x)$, but $T' \supseteq T \vdash \theta(\overline{n})$ for every $n \in \mathbb{N}$]." (p. 179).

" [46]Of course, the existence of classes [T] that are consistent but not ω-consistent is thus proved only on the assumption that there exists some consistent [T] (that is, that [*Principia mathematica*] is consistent)." (p. 179).

$T + \neg G$ is a counterexample to many things.

This theory is Σ_1-complete, but not Σ_1-sound.

It has also *false* Gödelian (Π_1-)sentences.

THE LAST PAGE OF GÖDEL (1931).

> "[W]e can, [in the 1st incompleteness theorem], replace the assumption of ω-consistency by the following: The proposition "[T] is inconsistent" is not [T]-PROVABLE. (Note that there are consistent [T] for which this proposition is [T]-PROVABLE.)" (p. 195).

This is because $T \vdash Con_T \rightarrow G$ [1] (thus $T \nvdash \neg G$ if $T \nvdash \neg Con_T$) from which also the 2nd theorem ($T \nvdash Con_T$) follows.

> "The results will be stated and proved in full generality in a sequel to be published soon [...]. In that paper, also, the proof of [the 2nd incompleteness theorem], only sketched here, will be given in detail." (p. 195).

Part II never appeared; one reason being probably the "prompt acceptance of" the results.

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[1]Formalized incompleteness: if $T \vdash G \leftrightarrow \neg Prov_T(\ulcorner G \urcorner)$, then $T \vdash Con_T \rightarrow G$.

AN INTERLUDE.

Theorem (3.4 in: Lajevardi & S. 2019)

The consistency of $T + \mathit{Con}_T$ ($T \nvdash \neg\mathit{Con}_T$) *is (also) a necessary and sufficient condition for the TRUTH (as well as the independence) of all the Gödelian* Π_1*-sentences.*

Proof.

If $T \vdash \neg\mathit{Con}_T$, then $T \vdash \mathit{Prov}_T(\ulcorner\bot\urcorner)$, so $T \vdash \bot \leftrightarrow \neg\mathit{Prov}_T(\ulcorner\bot\urcorner)$!
If $T \vdash \gamma \leftrightarrow \neg\mathit{Prov}_T(\bar{\gamma})$, $\gamma \in \Pi_1$, but $\mathbb{N} \vDash \neg\gamma$, then $T \vdash \neg\gamma$ (by the Σ_1-completeness), and so $T \vdash \neg\mathit{Con}_T$ (by $T \vdash \mathit{Con}_T \rightarrow \gamma$). [1] ■

- DANIEL ISAACSON (2011); "Necessary and Sufficient Conditions for Undecidability of the Gödel Sentence and Its Truth", in: *Logic, Mathematics, Philosophy, Vintage Enthusiasms*, Springer, 135–152.

ω-Consistency $\not\dashv\vdash$ Consistency with Con_T $\not\dashv\vdash$ Simple Consistency

The Question Remains.

Question:

Why is ω-Consistency *purely formal and much weaker* than Soundness?

We know that "ω-consistency" is FORMALLY DEFINABLE:

$\omega\text{-}Con_T \equiv \neg\exists\chi(v)\colon Prov_T(\ulcorner\neg\forall v\chi(v)\urcorner) \wedge \forall w\, Prov_T(\ulcorner\chi(\bar{w}/v)\urcorner)$

syntactic notions (variables, terms, numerals, formulas, proofs) are definable

For Gödel (1931): $Con_T \equiv \exists x\colon Formula(x) \wedge \neg Prov_T(x)$.

While "soundness" is *not* FORMALLY DEFINABLE!

Soundness implies ω-Consistency. But NOT the converse!

Could Gödel have meant: "to replace the second of the assumptions just mentioned by a purely formal and (thus) much weaker one"?

THE NECESSITY OF ω–CONSISTENCY.

- KURT GÖDEL (1931).

  "we can ... replace the assumption of ω-consistency by the following: The proposition "[T] is inconsistent" is not [T]-PROVABLE." (p. 195).

- BARKLEY ROSSER (1936); Extensions of Some Theorems of Gödel and Church, *The Journal of Symbolic Logic* 1(3):87–91.

  "... a modification is made in Gödel's proofs of his theorems ... it is proved that simple consistency implies the existence of undecidable propositions ..." (p. 87).

THE POINT OF ω–CONSISTENCY.

- CRAIG SMORYŃSKI (1985);
  Self-Reference and Modal Logic, Springer.

  "Remark: One weakness of Gödel's original work was his introduction of the semantic notion of ω-consistency. I find this notion to be pointless, but I admit many proof theorists take it seriously. ... I once showed" [that this notion is equivalent to some reflection principle]. (p. 158).

pointless: maybe, semantic: never!

AN EQUIVALENT OF ω–CONSISTENCY.

- PETER MILNE (2007); On Gödel Sentences and What They Say, *Philosophia Mathematica* 15(2):193–226.

  "LEMMA 1. *ω-consistency of T is both necessary and sufficient to ensure that, for each predicate $\pi(x, y)$ representing* [that y is a T-proof of x], *the set of sentences $T \cup \{\exists y \pi(\ulcorner\alpha\urcorner, y) \to \alpha : \alpha \in \mathit{Sentences}\}$ is consistent.*" (p. 203).

Some Semantic Aspects of ω–Consistency.

How Much Soundness Does (ω-)Consistency Have/Preserve?

- Consistency $\Longrightarrow$ Π_1-Soundness (by Σ_1-completeness)
- ω-Consistency $\Longrightarrow$ Π_3-Soundness

Isaacson (2011, Theorem 17) noting that $\Sigma_2\text{-Sound}_T \equiv \Pi_3\text{-Sound}_T$.

- Con_T & $\sigma \in \Sigma_1\text{-Th}_{\mathbb{N}} \Longrightarrow Con_{T+\sigma}$ (by Σ_1-completeness)
- $\omega\text{-}Con_T$ & $\boldsymbol{\sigma} \in \Sigma_3\text{-Th}_{\mathbb{N}} \Longrightarrow \omega\text{-}Con_{T+\boldsymbol{\sigma}}$

In Isaacson (2011, Theorem 22) this is attributed to George Kreisel (2005) for $\boldsymbol{\sigma} \in \Pi_1$. Works for $\boldsymbol{\sigma} \in \Pi_2$ and $\boldsymbol{\sigma} \in \Sigma_3$ too.

WEAKNESS OF ω–CONSISTENCY.

- LEON HENKIN (1954); A Generalization of the Concept of ω-Consistency, *The Journal of Symbolic Logic* 19(3):183–196.
- GEORG KREISEL, Mathematical Reviews (MR63324) 1955.

Attributed to KREISEL (1955) in ISAACSON (2011, Proposition 19):

Proof.

Let ω-Con$_T$, and put $K \in \Sigma_3$ satisfy $\mathbb{P} \vdash K \leftrightarrow \neg\omega\text{-}Con_T(\ulcorner K \urcorner)$.

Note that $\omega\text{-}Con_T(x) \in \Pi_3$.[2]

If $\mathbb{N} \vDash K$, then $\mathbb{N} \vDash \neg\omega\text{-}Con_{T+K}$, contrary to what was shown above!

So, $\mathbb{N} \nvDash K$, and $T+K$ is an ω-consistent but (Σ_3-)*un*sound theory! ■

The classic proof for Undefinability of Truth; just put $\omega\text{-}Con_T(x)$ in the place of $Tr(x)$. KREISEL's K is the LIAR's sentence for it.

[2] $\omega\text{-}Con_T(x) \equiv \forall\chi(v)\,[\![\neg Prov_{T+x}(\ulcorner\neg\forall v\chi(v)\urcorner) \vee \exists w\,\neg Prov_{T+x}(\ulcorner\chi(\bar{w}/v)\urcorner)]\!]$

More Semantic Aspects of ω–Consistency.

How Much Soundness Does (ω-)Consistency Have Exactly?

- Consistency $\Longrightarrow$ Π_1-Soundness
  Consistency $\not\Longrightarrow$ Σ_1-Soundness

$U = T + \neg G$

- ω-Consistency $\Longrightarrow$ Π_3-Soundness
  ω-Consistency $\not\Longrightarrow$ Σ_3-Soundness

$U = T + K$

YET MORE SEMANTIC ASPECTS OF ω–CONSISTENCY.

How Much Truth Does (ω-)Consistency Preserve Exactly?

- $\mathit{Con}_T \ \& \ \sigma \in \Sigma_1\text{-Th}_{\mathbb{N}} \Longrightarrow \mathit{Con}_{T+\sigma}$ (by Σ_1-completeness)
  $\mathit{Con}_U \ \& \ \pi \in \Pi_1\text{-Th}_{\mathbb{N}} \not\Longrightarrow \mathit{Con}_{U+\pi}$
  Put $U = T + \neg\mathit{Con}_T$, and $\pi = \mathit{Con}_U$. Then $\mathbb{N} \vDash \pi$ and $U \vdash \neg\pi$.

- $\omega\text{-}\mathit{Con}_T \ \& \ \boldsymbol{\sigma} \in \Sigma_3\text{-Th}_{\mathbb{N}} \Longrightarrow \omega\text{-}\mathit{Con}_{T+\boldsymbol{\sigma}}$
  $\omega\text{-}\mathit{Con}_U \ \& \ \boldsymbol{\pi} \in \Pi_3\text{-Th}_{\mathbb{N}} \not\Longrightarrow \mathit{Con}_{U+\boldsymbol{\pi}}$[3]
  Put $U = T + K$, and $\pi = \neg K$. Then $\pi \in \Pi_3\text{-Th}_{\mathbb{N}}$ and $U \vdash \neg\pi$.

[3]This is not a misprint: adding a true Π_3-sentence to an ω-consistent theory may not even yield a consistent theory!

Some Syntactic Aspects of ω–Consistency.

- $Con_T \Longrightarrow \forall\psi$: $Con_{T+\psi} \vee Con_{T+\neg\psi}$ Lindenbaum's Lemma
- ω-$Con_T \Longrightarrow \forall\psi$: ω-$Con_{T+\psi} \vee$ ω-$Con_{T+\neg\psi}$

  Isaacson (2011, Theorem 21)

$\mathrm{Lim}_{\subseteq}\,\mathrm{Con} = Con$, but $\mathrm{Lim}_{\subseteq}\,\omega\text{-}\mathrm{Con} \neq \omega\text{-}Con$.

- ω-$Con_T \wedge Complete_T \Longrightarrow T = \mathrm{Th}_{\mathbb{N}}$

  Isaacson (2011, Theorem 20)
- $Con_T \wedge Complete_T \not\Longrightarrow T = \mathrm{Th}_{\mathbb{N}}$

  Put $\mathcal{M} \vDash S + \neg Con_S$ with $\mathbb{N} \vDash Con_S$, and $T = \mathrm{Th}_{\mathcal{M}}$.

MORE SYNTACTIC ASPECTS OF ω–CONSISTENCY.

- $\mathit{Con}_T \Longrightarrow \mathit{Con}_T(\text{“}\neg\mathit{Con}_T\text{”})$ GÖDEL's 2nd Theorem
- $\omega\text{-}\mathit{Con}_T \Longrightarrow \omega\text{-}\mathit{Con}_T(\text{“}\neg\omega\text{-}\mathit{Con}_T\text{”})$ $\mathbb{G}_2(\omega\text{-}\mathit{Con})$

  GEORGE BOOLOS (1993, page xxxi) [*The Logic of Provability*, Cambridge University Press] says that this follows from BARKLEY ROSSER (1937) [Gödel Theorems for Non-Constructive Logics, *J. Symbolic Logic* 2(3):129–137].
- $\mathit{Con}_T \Longrightarrow \exists\rho\colon \mathit{Con}_{T+\rho} \wedge \mathit{Con}_{T+\neg\rho}$

  ROSSER (1936, Theorem II)

⊛ $\omega\text{-}\mathit{Con}_T \Longrightarrow \exists\rho\colon \omega\text{-}\mathit{Con}_{T+\rho} \wedge \omega\text{-}\mathit{Con}_{T+\neg\rho}$??

If $\mathbb{N} \vDash T$, then $\rho = K$ works, since $\mathbb{N} \vDash T + \neg K$, so $\omega\text{-}\mathit{Con}_{T+\neg K}$.

If $T + \omega\text{-}\mathit{Con}_T$ is ω-consistent, then $\rho = \omega\text{-}\mathit{Con}_T$ by $\mathbb{G}_2(\omega\text{-}\mathit{Con})$.

If $T + \omega\text{-}\mathit{Con}_T$ is not ω-consistent, then ... ?!

Thank You!

Thanks to

The Participants .... For Listening ⋯

and

The Organizers, For Taking Care of Everything ⋯