

A Quick Introduction to MATHEMATICAL LOGIC

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The Halting Problem (1)

Some Recursive Functions may Never Halt (may not have outputs on some inputs); e.g.,

$$D(x, y) = [\mu z. z+x=y] = \begin{cases} y-x & \text{if } x \leq y \\ \text{undefined} & \text{if } x > y \end{cases}$$

halts only when $x \leq y$.

Notation: $\begin{cases} f(x) \downarrow & f \text{ is defined at } x \\ f(x) \uparrow & f \text{ is not defined at } x \end{cases}$

Recursive Functions can be encoded by natural numbers:
Any description (proof) of a recursive function is a well-built sequence of $\langle Z, S, \pi_j^k, A, M, E, \chi_{\leq}, \wp, \circ, \mu \rangle$ (\circ stands for composition) and thus can be coded in \mathbb{N} .

Denote the (Gödel) code of the recursive function f by $\ulcorner f \urcorner$.

The Halting Problem (2)

Theorem (Turing 1937)

There is no recursive function h such that for any Recursive f ,
 $h(\ulcorner f \urcorner) = 1 \iff f(\ulcorner f \urcorner) \downarrow$ *and* $h(\ulcorner f \urcorner) = 0 \iff f(\ulcorner f \urcorner) \uparrow$.

Proof.

Otherwise, $g(x) = \mu z. (z + h(x) = z)$ would be recursive too, for which we have $g(\ulcorner f \urcorner) \downarrow \iff f(\ulcorner f \urcorner) \uparrow$ for every recursive f . Putting $f = g$ we get the contradiction $g(\ulcorner g \urcorner) \downarrow \iff g(\ulcorner g \urcorner) \uparrow$! ■

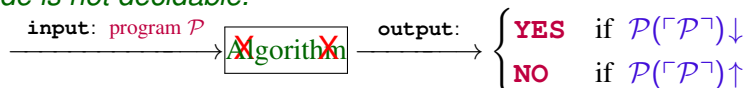
Corollary

There is no algorithmic way for recognizing whether a given program is a virus (self-generating) or not.

An Undecidable, and a Non-Enumerable Set

Corollary (The Halting Set is *Not* Decidable)

The set of all (single-input) programs which halt on their own code is not decidable.



Theorem (The Halting Set *Is* Enumerable)

An input-free algorithm enumerates the set $\{\mathcal{P} \mid \mathcal{P}(\ulcorner \mathcal{P} \urcorner) \downarrow\}$.

Proof.

Enumerate all the (single-input) programs $\mathcal{P}_0, \mathcal{P}_1, \dots$.

Let $n := 1$; for $i = 0$ to $i = n$ run the n stages of $\mathcal{P}_i(\ulcorner \mathcal{P}_i \urcorner)$; if it halts then PRINT " i "; let $n := n + 1$ and repeat. ■

Corollary (The Non-Halting Set is *Not* Enumerable)

The set $\{\mathcal{P} \mid \mathcal{P}(\ulcorner \mathcal{P} \urcorner) \uparrow\}$ is not enumerable.

Decidable Structures

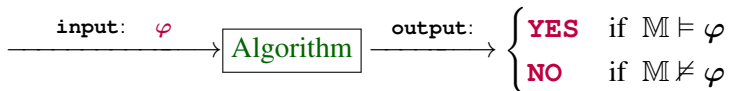
Definition (Decision Problem for a Structure)

Fix a structure $\langle \mathbb{M}; \mathcal{L} \rangle$.

Input: a first-order \mathcal{L} -sentence φ . Output: $\begin{cases} \text{YES} & \text{if } \mathbb{M} \models \varphi \\ \text{NO} & \text{if } \mathbb{M} \not\models \varphi \end{cases}$

Definition (Decidable Structure)

A structure is decidable if its decision problem is algorithmically solvable.



Enumerability in Structures

Algorithm $\xrightarrow{\text{output:}}$ $\{\varphi_0, \varphi_1, \varphi_2, \dots\} = \{\varphi \mid \mathbb{M} \models \varphi\}$

Theorem (Enumerable Structures are Decidable)

If \mathbb{M} is an enumerable structure, then it is decidable.

Proof.

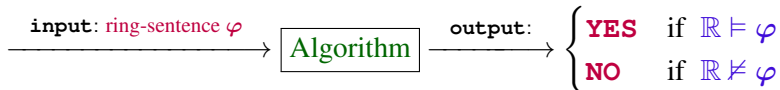
If $\{\varphi \mid \mathbb{M} \models \varphi\}$ is enumerable, then so is its complement $\{\psi \mid \mathbb{M} \not\models \psi\}$ because $\{\psi \mid \mathbb{M} \not\models \psi\} = \{\psi \mid \mathbb{M} \models \neg\psi\}$. ■

input: φ \rightarrow Algorithm $\xrightarrow{\text{output:}}$ $\begin{cases} \text{YES} & \text{if } \mathbb{M} \models \varphi \\ \text{NO} & \text{if } \mathbb{M} \not\models \varphi \end{cases}$

Tarski's Theorems

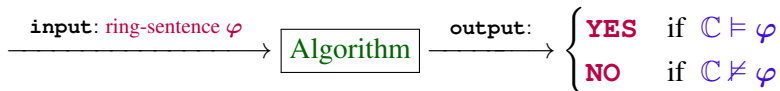
Theorem (Decidability of the Real (Ordered) Field)

The structure $\langle \mathbb{R}; 0, 1, -, \iota', +, \times, \leq \rangle$ is decidable.



Theorem (Decidability of the Complex Field)

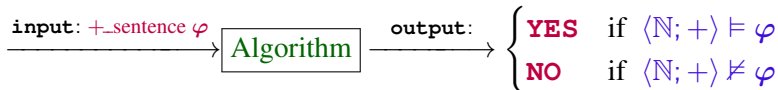
The structure $\langle \mathbb{C}; 0, 1, -, \iota', +, \times \rangle$ is decidable.



Arithmetics of Presburger and Skolem

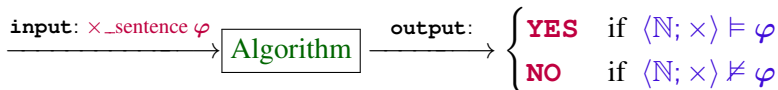
Theorem (Presburger 1929)

The structure $\langle \mathbb{N}; 0, 1, +, \leq \rangle$ is decidable.



Theorem (Skolem 1930)

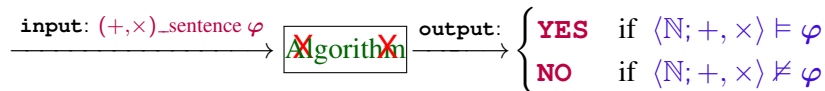
The structure $\langle \mathbb{N}; 0, 1, \times \rangle$ is decidable.



Full Arithmetic $\langle \mathbb{N}; +, \times \rangle$

Theorem (Gödel's Incompleteness 1931)

The structure $\langle \mathbb{N}; 0, 1, +, \times, \leq \rangle$ is *not* decidable.



Corollary

The structure $\langle \mathbb{Z}; 0, 1, -, +, \times, \leq \rangle$ is *undecidable* too.

Proof.

\mathbb{N} is definable in it by the formula $0 \leq x$. ■

THE END

Corollary (J. Robinson 1949)

*The structure $\langle \mathbb{Q}; 0, 1, -, i', +, \times, \leq \rangle$ is **undecidable** too.*

Corollary

*The structure $\langle \mathbb{C}; 0, 1, -, i', e^x, +, \times \rangle$ is **undecidable** too.*

Problem (Open — Tarski)

Is the Real Exponential Field $\langle \mathbb{R}; 0, 1, -, i', e^x, +, \times, \leq \rangle$ decidable or not?