A Quick Introduction to MATHEMATICAL LOGIC

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AMS Math Subject Classification (1970)

From (almost) 1970's AMS divided Mathematics into

- History and Foundations
  - 01. History and Biography
  - 02. Logic and Foundations

    ⋮
- Algebra
- Analysis
- Geometry

⋮

AMS Math Subject Classification of *Logic* (1970)

02 LOGIC AND FOUNDATIONS

- 02A PHILOSOPHICAL AND CRITICAL
- 02B CLASSICAL LOGICAL SYSTEMS
- 02C NONCLASSICAL FORMAL SYSTEMS
- 02D PROOF THEORY
- 02E CONSTRUCTIVE MATHEMATICS
- 02F RECURSION THEORY
- 02G METHODOLOGY OF DEDUCTIVE SYSTEMS
- 02H MODEL THEORY
- 02I —
- 02J ALGEBRAIC LOGIC
- 02K SET THEORY

AMS Math Subject Classification of *Logic* (1980)

From 1980 AMS divided Mathematics into

00. General
01. History and Biography
02. —
03. Mathematical Logic and Foundations
04. Set Theory

⋮

AMS Math Subject Classification of *Logic* (2000)

From 2000 AMS divided Mathematics into

00. General
01. History and Biography
02. —
03. Mathematical Logic and Foundations
04. —
05. Combinatorics

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AMS Math Subject Classification of *Logic* (2020)

03 Mathematical Logic and Foundations

- 03A Philosophical Aspects of Logic and Foundations
- 03B General Logic
- 03C Model Theory
- 03D Computability and Recursion Theory
- 02E Set Theory
- 03F Proof Theory and Constructive Mathematics
- 02G Algebraic Logic
- 02H Nonstandard Models

Foundations — Why?

Because everything is (can be) a set! Even numbers!

$0 \quad \emptyset \ (=\{\})$

$1 \quad \{\emptyset\}$

$2 \quad \{\emptyset, \{\emptyset\}\}$

$3 \quad \Big\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\Big\}$

$\vdots$

$n+1 \quad \{0, 1, \cdots, n\} = n \cup \{n\}$

$\vdots$

Foundations — How?

Graphs, Groups, Algebras, . . . everything is (can be defined as) a set; even ordered pairs:

Definition (Kuratowski 1921)
$\langle x, y\rangle = \Big\{\{x\}, \{x, y\}\Big\}$

Exercise: Show that $\langle x, y\rangle = \langle a, b\rangle \iff x = a \wedge y = b$.

A relation is a *set* of ordered pairs, a function is a relation . . .

Now we are used to the terminology of Set Theory after the wave of *New Mathematics* . . .

Sets for Counting

Numbers — Having the same number of things ...

Equinumerosity — Having a bijection between them ...

Surprises:

- $\mathbb{N} \cong \mathbb{N} - \{0\}$: $f(x) = x + 1$.
- $\mathbb{Z} \cong 2\mathbb{Z}$(Even Integers): $f(x) = 2x$.
- $\mathbb{N}^2 \cong \mathbb{N}$: $f(x, y) = 2^x(2y+1) - 1$.

$\vdots$

- but $\mathcal{P}(\mathbb{N}) \ncong \mathbb{N}$ since for every $f: \mathbb{N} \to \mathcal{P}(\mathbb{N})$, the set $D_f = \{n \mid n \notin f(n)\}$ is not in the range of f (if $D_f = f(m)$, then $m \in D_f \leftrightarrow m \notin \underline{f(m)} \leftrightarrow m \notin \underline{D_f}$!) remember the Liar's Paradox?

Crisis in the Foundations of Math.

Russell's Paradox

The set of all sets that are not members of themselves.

$$\mathfrak{R} = \{x \mid x \notin x\}, \quad \mathfrak{R} \in \mathfrak{R} \iff \mathfrak{R} \notin \mathfrak{R}!$$

So, the axiom of unrestricted comprehension is not valid! $\{x \mid \varphi(x)\}$

If $\mathbf{1} = \{\{a\} \mid a = a\}$, or $\{x \mid \exists y \forall z (z \in x \leftrightarrow z = y)\}$ $(x = \{y\})$, then let $\mathbf{1}' = \{x \in \mathbf{1} \mid \exists y (x = \{y\} \wedge x \notin y)\}$ $\left(\Big\{\{a\} \mid \{a\} \notin a\Big\}\right)$.

Now, $\{\mathbf{1}'\} \in \mathbf{1}' \leftrightarrow \exists y [\{\mathbf{1}'\} = \{y\} \wedge \{\mathbf{1}'\} \notin y] \leftrightarrow \{\mathbf{1}'\} \notin \mathbf{1}'!$

Some Exercises (1)

1. Write the syllogisms $\mathcal{S}\mathfrak{a}\mathcal{P}$, $\mathcal{S}\mathfrak{i}\mathcal{P}$, $\mathcal{S}\mathfrak{e}\mathcal{P}$, and $\mathcal{S}\mathfrak{o}\mathcal{P}$ in Predicate Logic by using the unary predicate symbols $\mathfrak{S}(x)$ and $\mathfrak{P}(x)$.

2. Prove the following in Group Theory:

$$\frac{}{\imath'(a*b)=\imath'(b)*\imath'(a)} \qquad \frac{x*x=\mathbf{e}}{a*b=b*a}$$

3. Prove that the following sentence, for any formula $\varphi(x)$, is true in every structure:

$$\exists x\big[\varphi(x)\rightarrow\forall y\,\varphi(y)\big]$$

Some Exercises (2)

1. Prove Barber's Paradox in Predicate Logic:

$$\neg\exists y \forall x \big[\theta(y,x) \leftrightarrow \neg\theta(x,x)\big]$$

2. Show that for every a, b, x, y we have

$$\Big\{\{x\},\{x,y\}\Big\} = \Big\{\{a\},\{a,b\}\Big\} \Longrightarrow x=a \ \wedge \ y=b$$

3. Prove that the mapping

$$\mathbb{N}^2 \to \mathbb{N}, \quad (x,y) \mapsto 2^x(2y+1)-1$$

is a bijection (1-1 and onto).