A Quick Introduction to Mathematical Logic

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Logic: A Way of Avoiding Mental Errors

- If $\mathscr{A} \longrightarrow \mathscr{B}$, and if $\mathscr{B}$, then can we infer that $\mathscr{A}$?

  If rain implies cloudiness, and it is cloudy, then will it rain?

- If $\mathscr{A} \longrightarrow \mathscr{B}$, and if $\mathscr{A}$, then can we infer that $\mathscr{B}$?

  If rain implies cloudiness, and it is raining, then is it cloudy also?

- If $\mathscr{A} \longrightarrow \mathscr{B}$, and if $\neg\mathscr{B}$, then can we infer that $\neg\mathscr{A}$?

  If rain implies cloudiness, and it is not cloudy, then is it not raining?

- If $\mathscr{A} \longrightarrow \mathscr{B}$, and if $\neg\mathscr{A}$, then can we infer that $\neg\mathscr{B}$?

  If rain implies cloudiness, and it is not raining, then is it not cloudy?

Truth Tables

p	q	$p \rightarrow q$	$\neg p$	$\neg q$
1	1	1	0	0
1	0	0	0	1
0	1	1	1	0
0	0	1	1	1

$0 \rightarrow X$ Ex Falso Quodlibet

Truth Tables . . . $\dfrac{\mathscr{A}\longrightarrow\mathscr{B},\ \mathscr{B}}{\mathscr{A}}$?

	$\mathscr{A}$	$\mathscr{B}$	$\mathscr{A}\longrightarrow\mathscr{B}$	$\neg\mathscr{A}$	$\neg\mathscr{B}$
▶	1	1	1	0	0
	1	0	0	0	1
▶	0	1	1	1	0
	0	0	1	1	1

$\mathscr{A}\longrightarrow\mathscr{B}$ and $\mathscr{B}$ do *not* imply $\mathscr{A}$; when $\mathscr{A}\equiv 0$ and $\mathscr{B}\equiv 1$.

$$\frac{\mathscr{A}\longrightarrow\mathscr{B},\ \mathscr{B}}{\therefore\quad\mathscr{A}}\ \text{X}$$

Truth Tables . . . $\dfrac{\mathscr{A}\rightarrow\mathscr{B},\;\mathscr{A}}{\mathscr{B}}$?

	$\mathscr{A}$	$\mathscr{B}$	$\mathscr{A}\rightarrow\mathscr{B}$	$\neg\mathscr{A}$	$\neg\mathscr{B}$
▶	1	1	1	0	0
	1	0	0	0	1
	0	1	1	1	0
	0	0	1	1	1

$\mathscr{A}\rightarrow\mathscr{B}$ and $\mathscr{A}$ do *always* imply $\mathscr{B}$.

$$\frac{\mathscr{A}\rightarrow\mathscr{B},\;\mathscr{A}}{\therefore\quad\mathscr{B}}$$ Modes Ponens

Truth Tables . . . $\dfrac{\mathscr{A} \rightarrow \mathscr{B},\ \ \neg\mathscr{B}}{\neg\mathscr{A}}$?

	$\mathscr{A}$	$\mathscr{B}$	$\mathscr{A} \rightarrow \mathscr{B}$	$\neg\mathscr{A}$	$\neg\mathscr{B}$
	1	1	1	0	0
	1	0	0	0	1
	0	1	1	1	0
▶	0	0	1	1	1

$\mathscr{A} \rightarrow \mathscr{B}$ and $\neg\mathscr{B}$ do *always* imply $\neg\mathscr{A}$.

$$\frac{\mathscr{A} \rightarrow \mathscr{B},\ \ \neg\mathscr{B}}{\therefore\quad \neg\mathscr{A}}$$ Modes Tollens

Truth Tables . . . $\dfrac{\mathscr{A} \rightarrow \mathscr{B},\ \neg\mathscr{A}}{\neg\mathscr{B}}$?

	$\mathscr{A}$	$\mathscr{B}$	$\mathscr{A} \rightarrow \mathscr{B}$	$\neg\mathscr{A}$	$\neg\mathscr{B}$
	1	1	1	0	0
	1	0	0	0	1
▶	0	1	1	1	0
▶	0	0	1	1	1

$\mathscr{A} \rightarrow \mathscr{B}$ and $\neg\mathscr{A}$ do *not* imply $\neg\mathscr{B}$; when $\mathscr{A} \equiv 0$ and $\mathscr{B} \equiv 1$.

$$\frac{\mathscr{A} \rightarrow \mathscr{B},\ \neg\mathscr{A}}{\therefore\quad \neg\mathscr{B}} \text{X}$$

A Puzzle

P says that "*Q is lying*", and Q says that "*both P and Q tell the truth*".

Who is lying and who tells the truth?

▶ P says $\neg Q$ Q says $P \wedge Q$ ◀

	P	Q	$\neg Q$	$P \wedge Q$	
	1	1	0	1	◀
▶	1	0	1	0	◀
▶	0	1	0	0	
	0	0	1	0	◀

(P says $\neg Q$) and (Q says $P \wedge Q$) imply that

P says THE TRUTH and Q LIES!

Another Puzzle

P says that "*either P or Q tells the truth*", and
Q says that "*P tells the truth if and only if Q does so*".
Who is lying and who tells the truth?

▶ P says $P \vee Q$ $\qquad$ Q says $P \leftrightarrow Q$ ◀

	P	Q	$P \vee Q$	$P \leftrightarrow Q$	
▶	1	1	1	1	◀
▶	1	0	1	0	◀
	0	1	1	0	
▶	0	0	0	1	

(P says $P \vee Q$) and (Q says $P \leftrightarrow Q$) imply that
P says THE TRUTH and Q ??!

A Paradox

What if somebody says that “*I am lying*”?!

- L says $\neg L$

	L	$\neg L$	$L \leftrightarrow \neg L$	$\neg(L \leftrightarrow \neg L)$	
	1	0	0	1	
	0	1	0	1	

We will come back to this later!

Conclusion

Truth Tables, however simple or boring they may seem, are still the best, and the most efficient, tools for verifying the truth (or falsity) of propositional sentences.

https://web.stanford.edu/class/cs103/tools/truth-table-tool/

$p \leftrightarrow \neg\neg p$

$p \rightarrow p \vee q$ $\qquad$ $q \rightarrow p \vee q$

$p \wedge q \rightarrow p$ $\qquad$ $p \wedge q \rightarrow q$

Some Exercises

Check that the following are tautologies (always true):

(AX$_1$) $\alpha \to (\beta \to \alpha)$

(AX$_2$) $[\alpha \to (\beta \to \gamma)] \to [(\alpha \to \beta) \to (\alpha \to \gamma)]$

(AX$_3$) $(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$

Prove

$$\alpha \to \alpha$$

by using (AX$_1$), (AX$_2$), (AX$_3$), and the Modus Ponens rule:

$$\frac{\mathscr{A} \longrightarrow \mathscr{B},\ \mathscr{A}}{\therefore \quad \mathscr{B}}$$