ROSSER PHENOMENON: Applications in Recursion Theory

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Rosser's Trick

`http://en.wikipedia.org/wiki/Rosser's_trick`

In mathematical logic, Rosser's trick is a method for proving Gödel's incompleteness theorems without the assumption that the theory being considered is ω–consistent (...). This method was introduced by J. Barkley Rosser in 1936, as an improvement of Gödel's original proof of the incompleteness theorems that was published in 1931.

While Gödel's original proof uses a sentence that says (informally) "This sentence is not provable", Rosser's trick uses a formula that says "If this sentence is provable, there is a shorter proof of its negation".

J. Barkley Rosser

http://en.wikipedia.org/wiki/J._Barkley_Rosser

John Barkley Rosser Sr. (December 6, 1907 – September 5, 1989) was an American logician, a student of Alonzo Church, and known for his part in the Church–Rosser theorem, in lambda calculus. He also developed what is now called the Rosser sieve, in number theory. He was later director of the Army Mathematics Research Center at the University of Wisconsin–Madison. Rosser wrote mathematical textbooks as well.

In 1936, he proved Rosser's trick, a stronger version of Gödel's first incompleteness theorem which shows that the requirement for ω–consistency may be weakened to consistency. Rather than using the liar paradox sentence equivalent to “I am not provable”, he used a sentence that stated “For every proof of me, there is a shorter proof of my negation”.

In prime number theory, he proved Rosser's theorem. The Kleene–Rosser paradox showed that the original lambda calculus was inconsistent.

The Trick

For every proof of me, there is a shorter proof of my negation.

Number x is a Rosser proof of the formula y in theory T:

$$\mathrm{Proof}^{\mathsf{R}}_T(x,y) \equiv \mathrm{Proof}_T(x,y) \wedge \neg\exists z\leqslant x\ \mathrm{Proof}_T(z,\mathtt{neg}(y))$$

The Rosser Sentence: $\rho \iff \neg\exists x\ \mathrm{Proof}^{\mathsf{R}}_T(x,\ulcorner\rho\urcorner)$

$$\rho \iff \forall x\ [\mathrm{Proof}_T(x,\ulcorner\rho\urcorner) \rightarrow \exists z\leqslant x\ \mathrm{Proof}_T(z,\ulcorner\neg\rho\urcorner)]$$

Rosser Consistency:

$\mathrm{Con}^{\mathsf{R}}(T) \equiv \forall x\ [\mathrm{Proof}_T(x,\bot) \rightarrow \exists z\leqslant x\ \mathrm{Proof}_T(z,\top)]$

but then $T\vdash \mathrm{Con}^{\mathsf{R}}(T)$! Contradictory with Gödel's 2nd Thm.

Rosser's Proof

If T is Σ_1-Complete and Consistent then $T \nvdash \rho, \neg\rho$

$$T \vdash \rho \iff \forall x\, [\mathrm{Proof}_T(x, \ulcorner \rho \urcorner) \rightarrow \exists z \leqslant x\, \mathrm{Proof}_T(z, \ulcorner \neg\rho \urcorner)]$$

If $T \vdash \rho$ then for some $n \in \mathbb{N}$, $\mathrm{Proof}_T(\underline{n}, \ulcorner \rho \urcorner)$ and so by Σ_1-completeness, $T \vdash \mathrm{Proof}_T(\underline{n}, \ulcorner \rho \urcorner)$. By the definition of ρ, $T \vdash \exists z \leqslant \underline{n}\, \mathrm{Proof}_T(z, \ulcorner \neg\rho \urcorner)$. So, for some $m \leqslant n$, $\mathrm{Proof}_T(m, \ulcorner \neg\rho \urcorner)$ whence $T \vdash \neg\rho$; contradiction!

If $T \vdash \neg\rho$ then $T \vdash \exists x [\mathrm{Proof}_T(x, \ulcorner \rho \urcorner) \wedge \forall z \leqslant x \neg \mathrm{Proof}_T(z, \ulcorner \neg\rho \urcorner)]$, also $T \vdash \mathrm{Proof}_T(\underline{n}, \ulcorner \neg\rho \urcorner)$ for some $n \in \mathbb{N}$.
So $x \leqslant n$ ($T \vdash n \leqslant x \vee x \leqslant n$). But then $T \vdash \bigvee_{i \leqslant n} \mathrm{Proof}_T(\underline{i}, \ulcorner \rho \urcorner)$, and so $T \vdash \rho$; contradiction!

Reduction Principle

Every Two $\exists-$Sets (RE sets) Can Be Separated

Let $A=\{u \mid \exists x\,\theta(u,x)\}$ and $B=\{u \mid \exists x\,\eta(u,x)\}$. Then for

$$A'=\{u \mid \exists x[\theta(u,x)\wedge\forall z\leqslant x\neg\eta(u,z)]\}$$

$$B'=\{u \mid \exists x[\eta(u,x)\wedge\forall z< x\neg\theta(u,z)]\}$$

we have $A'\subseteq A,\ B'\subseteq B,\ A'\cap B'=\emptyset,\ A'\cup B'=A\cup B$.

For RE sets $W_e=\{u \mid \exists z\,\mathbf{T}(e,u,z)\}$ let

$$W_i\preccurlyeq W_j=\{u \mid \exists z[\mathbf{T}(i,u,z)\wedge\forall y\leqslant z\neg\mathbf{T}(j,u,y)]\},$$
$$W_i\prec W_j=\{u \mid \exists z[\mathbf{T}(i,u,z)\wedge\forall y< z\neg\mathbf{T}(j,u,y)]\}.$$

Whence, $A'=W_i\preccurlyeq W_j$ and $B'=W_j\prec W_i$ will do.

If $W_i\cup W_j=\mathbb{N}$ Then $W_i\preccurlyeq W_j$ and $W_j\prec W_i$ are Recursive!

Can All RE Sets Be Separated Recursively?

Recursively Inseparable Sets

A and B are said to be *recursively inseparable* when there exists no `recursive` set C such that $A \subseteq C$ and $B \cap C = \emptyset$.

Example: $A = \{u \mid \varphi_u(u) = 0\}$ and $B = \{u \mid \varphi_u(u) \neq 0\}$.
If $A \subseteq C$ and $B \cap C = \emptyset$ and $\chi_C = \varphi_e$, then

$$e \in C \Longrightarrow \varphi_e(e) = 1 \Longrightarrow e \in B \Longrightarrow e \notin C$$

$$e \notin C \Longrightarrow \varphi_e(e) = 0 \Longrightarrow e \in A \Longrightarrow e \in C$$

So, $e \in C \longleftrightarrow e \notin C$; contradiction!

RE and Undecidable Sets

RE and Recursively Inseparable Sets

Resembles $\overline{\mathbf{K}} = \{u \mid \varphi_u(u)\uparrow\} = \{u \mid u \notin W_u\}$.

If $\overline{\mathbf{K}} = W_\alpha$ then $\alpha \in \overline{\mathbf{K}} \longleftrightarrow \alpha \notin \overline{\mathbf{K}}$.

Which resembles $\mathcal{D}_F = \{a \in A \mid a \notin F(a)\}$ for $F : A \to \mathscr{P}(A)$.

If $\mathcal{D}_F = F(\alpha)$ then $\alpha \in \mathcal{D}_F \longleftrightarrow \alpha \notin \mathcal{D}_F$.

Similar to Russel's Paradox:

$\mathcal{R} = \{x \mid x \notin x\}$ implies $\mathcal{R} \in \mathcal{R} \longleftrightarrow \mathcal{R} \notin \mathcal{R}$.

More Examples of Rec. Insep. RE Sets:

$A_i = \{u \mid \varphi_u(u) = i\}$ and $A_j = \{u \mid \varphi_u(u) = j\}$ for $i \neq j$.

$A_i = \{u \mid \varphi_u(0) = i\}$ and $A_j = \{u \mid \varphi_u(0) = j\}$ for $i \neq j$.

Existence of an RE but Undecidable Set

Beginning of Theoretical Computer Science

$\mathbf{K} = \{u \mid u \in W_u\}$ is RE and Undecidable.

Many more examples of $\Sigma_1 - \Delta_1$:

$\{u \mid \varphi_u(0)\downarrow\}$ $\{u \mid \exists x\, \varphi_u(x)\downarrow\}$

$\{u \mid \varphi_u(0) = 0\}$ $\{u \mid \exists x\, \varphi_u(x) = 0\}$

$\{u \mid \varphi_u(u) = 0\}$ $\{u \mid \varphi_u(u)\downarrow \neq 0\}$

RICE'S THEOREM: Every Non-Trivial Index Set is Undecidable.
For any $A \in \Sigma_1$ where $\emptyset \neq A \neq \mathbb{N}$ and
$\varphi_x = \varphi_y \Longrightarrow (x \in A \leftrightarrow y \in A)$, we have $A \in \Sigma_1 - \Delta_1$.

Many More Examples of Recursively Inseparable Sets

Rice's Theorem

Any Two Non-Empty Disjoint Index Sets A and B Are Recursively Inseparable.

Proof: For $a \in A$ and $b \in B$, if $A \subseteq C \subseteq \overline{B}$ and $\chi_C = \varphi_e$, then let

$$\varphi_n(z) = \begin{cases} \varphi_a(z) & \text{if } \varphi_e(n)\downarrow = 0 \\ \varphi_b(z) & \text{if } \varphi_e(n)\downarrow \neq 0 \\ \uparrow & \text{otherwise (if } \varphi_e(n)\uparrow) \end{cases}$$

$$n \in C \Rightarrow \varphi_e(n) = 1 \Rightarrow \varphi_n = \varphi_b \Rightarrow n \in B \Rightarrow n \notin C$$
$$n \notin C \Rightarrow \varphi_e(n) = 0 \Rightarrow \varphi_n = \varphi_a \Rightarrow n \in A \Rightarrow n \in C$$

So, $n \in C \longleftrightarrow n \notin C$, contradiction!

Connections with Incompleteness

Gödel's First Theorem $\equiv$ Existence of a $\Sigma_1 - \Delta_1$ Set

For a $\Sigma_1 - \Delta_1$ Set A and a Σ_1 Sound Theory T,

$$\Sigma_1 \ni \{a \mid T \vdash \text{“}a \in \overline{A}\text{”}\} \subseteq \overline{A} \notin \Sigma_1.$$

So, there exists some $\alpha \in \overline{A}$ such that $T \nvdash \text{“}\alpha \in \overline{A}\text{”}$!

This argument shows the incompleteness of any Σ_1-complete and $\omega-$consistent theory.

Another example of a $\Sigma_1 - \Delta_1$ set:

$\{\ulcorner\theta\urcorner \mid T \vdash \theta\}$ for an RE and sound theory T

like $\mathbf{ZFC}$ or $\mathbf{PA}$ or $\mathbf{I\Sigma_n}$ or $\mathbf{I_{open}}$ or $\mathbf{Q}$ ...

$\triangleright$ Every Δ_1 Theory Can Be Completed To a Δ_1 Theory.

Connections with Incompleteness

Incompleteness Theorem of Gödel–Rosser

Rosser's Trick shows that the sets of

$T-$Derivable Sentences $\mathtt{A} = \{\ulcorner\theta\urcorner \mid T \vdash \theta\}$

and $T-$Refutable Sentences $\mathtt{B} = \{\ulcorner\theta\urcorner \mid T \vdash \neg\theta\}$

are Recursively Inseparable.

H.B. ENDERTON, *A Mathematical Introduction to Logic*, 2nd ed. Acad. Pres. 2001. ex. 1 p. 245

If $C \in \Delta_1$ satisfies $\mathtt{A} \subseteq C \subseteq \overline{\mathtt{B}}$ and

$C = \{u \mid \exists y\, \theta(y,u)\}$, $\overline{C} = \{u \mid \exists y\, \eta(y,u)\}$ then let

$$T \vdash \sigma \longleftrightarrow \forall y[\theta(y,\ulcorner\sigma\urcorner) \rightarrow \exists z \leqslant y\; \eta(z,\ulcorner\sigma\urcorner)]$$

One Can Show $\ulcorner\sigma\urcorner \in C \iff \ulcorner\sigma\urcorner \in \overline{C}$, Contradiction!

Connections with Incompleteness

Gödel–Rosser Theorem $\Longrightarrow$ $\exists$ Recursively Inseparable RE Sets

$$\mathtt{A}=\{\ulcorner\theta\urcorner \mid T\vdash\theta\}\subseteq C=\{u \mid \exists y\,\theta(y,u)\}$$

$$\mathtt{B}=\{\ulcorner\theta\urcorner \mid T\vdash\neg\theta\}\subseteq \overline{C}=\{u \mid \exists y\,\eta(y,u)\}$$

$$T\vdash\sigma\longleftrightarrow\forall y[\theta(y,\ulcorner\sigma\urcorner)\rightarrow\exists z\leqslant y\;\eta(z,\ulcorner\sigma\urcorner)]$$

(1) $\big[\ulcorner\sigma\urcorner\in C\big]\Rightarrow\big[\exists n\in\mathbb{N}\,\theta(n,\ulcorner\sigma\urcorner)\bigwedge\forall m\in\mathbb{N}\,\neg\eta(m,\ulcorner\sigma\urcorner)\big]\Rightarrow$
$\big[T\vdash\exists y[=n]\big(\theta(y,\ulcorner\sigma\urcorner)\wedge\forall z\leqslant y\neg\eta(z,\ulcorner\sigma\urcorner)\big)\big]\Rightarrow$
$\big[T\vdash\neg\sigma\big]\Rightarrow\big[\ulcorner\sigma\urcorner\in\mathtt{B}\big]\Rightarrow\big[\ulcorner\sigma\urcorner\in\overline{C}\big]$

(2) $\big[\ulcorner\sigma\urcorner\in\overline{C}\big]\Rightarrow\big[\exists n\in\mathbb{N}\,\eta(n,\ulcorner\sigma\urcorner)\bigwedge\forall m\in\mathbb{N}\,\neg\theta(m,\ulcorner\sigma\urcorner)\big]\Rightarrow$
$\big[T\vdash\forall y\leqslant n(\neg\theta(y,\ulcorner\sigma\urcorner))\Rightarrow[\theta(y,\ulcorner\sigma\urcorner)\rightarrow\exists z\leqslant y\;\eta(z,\ulcorner\sigma\urcorner)]\big]$
$\big[T\vdash\forall y\geqslant n(\exists z\leqslant y\;\eta(z,\ulcorner\sigma\urcorner))\big]\Rightarrow$
$\big[T\vdash\forall y\geqslant n[\theta(y,\ulcorner\sigma\urcorner)\rightarrow\exists z\leqslant y\;\eta(z,\ulcorner\sigma\urcorner)]\big]\Rightarrow$
$\big[T\vdash\sigma\big]\Rightarrow\big[\ulcorner\sigma\urcorner\in\mathtt{A}\big]\Rightarrow\big[\ulcorner\sigma\urcorner\in C\big]$.

Thus, $\ulcorner\sigma\urcorner\in C\iff\ulcorner\sigma\urcorner\in\overline{C}$; Contradiction!

Connections with Incompleteness

$\exists$ Recursively Inseparable RE Sets $\Longrightarrow$ Gödel–Rosser Theorem

D. VAN DALEN, *Logic and Structure*, 5nd ed. Springer (1980–)2013. Th. 8.7.10 (Undec. **PA**)

If A and B are REC. INSEP. RE $\&$ T is Σ_1–complete such that

$$T \vdash x \in B \rightarrow x \notin A \quad \text{or} \quad T \vdash x \in A \rightarrow x \notin B$$

Then $A \subseteq \{u \mid T \vdash \text{“}u \in A\text{”}\} \subseteq \{u \mid T \vdash \text{“}u \notin B\text{”}\} = T_B \in \Sigma_1$

and $B \subseteq \{u \mid T \vdash \text{“}u \in B\text{”}\} \subseteq \{u \mid T \vdash \text{“}u \notin A\text{”}\} = T_A \in \Sigma_1$

satisfy $A \cap T_A = \emptyset = B \cap T_B$.

If $T_A \cup T_B = \mathbb{N}$ then $T_B \preccurlyeq T_A$ separates A and B recursively!

$$\left[A \subseteq T_B \preccurlyeq T_A (\in \Sigma_1) = \overline{T_A \prec T_B} (\in \overline{\Sigma_1}) \subseteq \overline{B}\right]$$

So, $T_A \cup T_B \neq \mathbb{N}$. Put $k \notin T_A \cup T_B$.

$T \nvdash \text{“}\underline{k} \notin A\text{”}$, and $T \nvdash \text{“}\underline{k} \notin B\text{”}$ so $T \nvdash \text{“}\underline{k} \in A\text{”}$.

Whence, $T \nvdash \text{“}\underline{k} \notin A\text{”}$ and $T \nvdash \text{“}\underline{k} \in A\text{”}$; so T is incomplete.

Thank You!

Thanks To
The Participants
for Listening and for Your Patience!
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