Theoremizing Paradoxes

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Liar Paradox

This Sentence is not True.

For $L \longleftrightarrow \neg L$ we have

`Propositional Logic` $\vdash \neg(p \longleftrightarrow \neg p)$.

TARSKI'S THEOREM:

There Is No Formula $\mathscr{T}$ Such That $T \vdash \psi \leftrightarrow \mathscr{T}(\overline{\psi})$ For All Formulae ψ
For Some Encoding $\overline{\varphi}$ Of φ
And Sufficiently Strong Theory T In A Sufficiently Strong Language.

GÖDEL'S THEOREM:

This Sentence Is Not $T-$Provable Is Indeed True and Not $T-$Provable
For A Sufficiently Strong And Sound Theory T.

Russell's Paradox

The Set of All Those

Sets Which Are Not Members of Themselves

... Does Not Exist!

The Inconsistency Of The Comprehension Principle:

For Any Formula φ The Set $\{x \mid \varphi(x)\}$ Exists.

A Theorem In Set Theory:

Set Theory $\vdash$ There Is No Set Which Contains All Sets.

by $R = \{x \mid x \notin x\}$ we have

$$\texttt{Set Theory} \vdash \neg\exists y \forall x (x \in y \longleftrightarrow x \notin x).$$

Even More:

$$\texttt{First Order Logic} \vdash \neg\exists y \forall x \big[\mathcal{S}(x,y) \longleftrightarrow \neg\mathcal{S}(x,x)\big].$$

Russell's Popularization of his paradox:

Barber's Paradox

Shaves (Only) The Ones Who Cannot Shave Themselves.

This resembles also

Grelling–Nelson Paradox

"Heterological" Is Heterological If And Only If It Is Not!

So, some paradoxes turn to theorems in mathematics or logic.
Also, some theorems are called paradox in the literature.

Drinker Paradox http://en.wikipedia.org/wiki/Drinker_paradox

There is someone in the pub such that,
if he is drinking, everyone in the pub is drinking.

$$\texttt{First Order Logic} \vdash \exists y \forall x \big[\mathcal{D}(y) \longrightarrow \mathcal{D}(x)\big].$$

Indeed, also

$$\texttt{First Order Logic} \vdash \exists y \forall x \big[\mathcal{D}'(x) \longrightarrow \mathcal{D}'(y)\big].$$

$$\texttt{First Order Logic} \vdash \forall x \exists y \big[\mathcal{D}(y) \longrightarrow \mathcal{D}(x)\big].$$

$$\texttt{First Order Logic} \vdash \forall x \exists y \big[\mathcal{D}'(x) \longrightarrow \mathcal{D}'(y)\big].$$

$$\texttt{First Order Logic} \vdash \forall x \exists y \big[\mathcal{D}(x) \longleftrightarrow \mathcal{D}(y)\big].$$

but $\texttt{First Order Logic} \nvdash \exists y \forall x \big[\mathcal{D}(x) \longleftrightarrow \mathcal{D}(y)\big].$

SELF–REFERENCE or DIAGONAL

For the sequence $W_0, W_1, W_2, \cdots$ of subsets of $\mathbb{N}$ the subset $\{m \mid m \notin W_m\}$ of $\mathbb{N}$ is not in the list. For if $\{m \mid m \notin W_m\} = W_k$ then $k \in W_k \iff k \notin W_k$, contradiction!

For the sequence $\alpha_0, \alpha_1, \alpha_2, \cdots$ of 0's and 1's ($\in \{0,1\}^{\mathbb{N}}$), the sequence β defined by $\beta(i) = 1 - \alpha_i(i)$ is not equal to any of α_n's. For if $\beta = \alpha_m$ then $\alpha_m(m) = \beta(m) = 1 - \alpha_m(m)$, contradiction!

For any $F : A \to \mathscr{P}(A)$ the sub-set $D_F = \{x \in A \mid x \notin F(x)\}$ of A is not in the range of F. For if $D_F = F(a)$ then $a \in D_F \iff a \notin F(a) \iff a \notin D_F$, contradiction!

A New Paradox:

Yablo's Paradox $Y_1, Y_2, Y_3, \cdots$

Y_n is True if and only if All Y_k's for $k > n$ are Untrue.

- If some Y_m is true, then $Y_{m+1}, Y_{m+2}, Y_{m+3}, \cdots$ are all untrue. Whence Y_{m+1} is true and untrue at the same time!
- If all Y_k's are untrue, then $Y_0, Y_1, Y_2, \cdots$ are true!

Theoremizing:

$$\texttt{Second Order Logic} \vdash \forall x \exists y (x < y) \wedge \forall x, y, z (x < y < z \rightarrow x < z) \longrightarrow \neg \exists X \forall u \big[X(u) \longleftrightarrow \forall v > u \neg X(v) \big].$$

If $X(a)$, then for some $b > a$, $\neg X(b)$ and for all $v > b$ we have $v > a$ and so $\neg X(v)$ which implies $X(b)$, contradiction. So $\forall \alpha \neg X(\alpha)$ and in particular $\forall \alpha > a \neg X(\alpha)$, whence $X(a)$; contradiction!

Theoremizing:

$\texttt{First Order Logic} \vdash \forall x \exists y (x < y) \wedge \forall x, y, z (x < y < z \rightarrow x < z)$
$\longrightarrow \neg \forall u \big[\psi(u) \longleftrightarrow \forall v > u \, \neg \psi(v) \big].$

Or $\texttt{First Order Logic} \vdash$

$$\forall x \forall y (x \mathsf{R} \mathfrak{s}(x) \wedge [\mathfrak{s}(x) \mathsf{R} y \rightarrow x \mathsf{R} y]) \Longrightarrow \exists u \big(\mathcal{D}(u) \longleftrightarrow \forall v \big[u \mathsf{R} v \rightarrow \mathcal{D}(v) \big] \big).$$

Find the weakest (first–order) condition Ψ on R such that

$\texttt{Second Order Logic} \vdash$
$$\Psi(R) \Longrightarrow \neg \exists X \forall x \big[X(x) \longleftrightarrow \forall y [x R y \rightarrow \neg X(y)] \big].$$

Conjecture

The Second–Order Predicate (of R)

$$\neg \exists X \forall x \big[X(x) \longleftrightarrow \forall y [x R y \rightarrow \neg X(y)] \big]$$

Is Not First–Order.

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(Propositional) Linear Temporal Logic (LTL):

$\bigcirc$: Next **$\Box$: Always**

The Intended Model: $\langle \mathbb{N}, \Vdash \rangle$ where $\Vdash \subseteq \mathbb{N} \times \texttt{Atoms}$ can be extended to all formulas by:

- $n \Vdash \varphi \wedge \psi$ iff $n \Vdash \varphi$ and $n \Vdash \psi$
- $n \Vdash \neg\varphi$ iff $n \nVdash \varphi$
- $n \Vdash \bigcirc\varphi$ iff $(n+1) \Vdash \varphi$
- $n \Vdash \Box\varphi$ iff $m \Vdash \varphi$ for every $m \geq n$

An Example of a Law of LTL: $\Box\bigcirc\varphi \equiv \bigcirc\Box\varphi$

$n \Vdash \Box\bigcirc\varphi$ iff $\forall x \geq n\big[x \Vdash \bigcirc\varphi\big]$ iff $\forall x \geq n\big[(x+1) \Vdash \varphi\big]$
iff $\forall x \geq n+1\big[x \Vdash \varphi\big]$ iff $(n+1) \Vdash \Box\varphi$ iff $n \Vdash \bigcirc\Box\varphi$

(Propositional) Linear Temporal Logic:

$\bigcirc$: **Next** $\square$: **Always**

Another Law of LTL: $\bigcirc\neg\varphi \equiv \neg\bigcirc\varphi$

$n \Vdash \bigcirc\neg\varphi$ iff $(n+1) \Vdash \neg\varphi$ iff $(n+1) \nVdash \varphi$ iff $n \nVdash \bigcirc\varphi$ iff $n \Vdash \neg\bigcirc\varphi$

Whence, $\bigcirc\square\neg\varphi \;\equiv\; \square\bigcirc\neg\varphi \;\equiv\; \square\neg\bigcirc\varphi$

Yablo's Paradox: As A Theorem In LTL:

Theorem

$LTL \vdash \neg(\varphi \leftrightarrow \square\bigcirc\neg\varphi)$ for all formulae φ.

Theorem

$LTL \nvdash (\varphi \leftrightarrow \bigcirc\Box\neg\varphi)$ for any formula φ.

Proof:

Otherwise, if $LTL \vdash \phi \leftrightarrow \bigcirc\Box\neg\phi$ then:

- If $m \Vdash \phi$ for some m, then $m \Vdash \bigcirc\Box\neg\phi$ so $(m+1) \Vdash \Box\neg\phi$, hence $(m+i) \Vdash \neg\phi$ for all $i \geq 1$. In particular, $(m+1) \Vdash \neg\phi$ and $(m+j) \Vdash \neg\phi$ for all $j \geq 2$ which implies that $(m+2) \Vdash \Box\neg\phi$ or $(m+1) \Vdash \bigcirc\Box\neg\phi$ so $(m+1) \Vdash \phi$, contradiction!
- So, $k \Vdash \neg\phi$ for all k, and so $k \Vdash \bigcirc\neg\Box\neg\phi$ thus $(k+1) \Vdash \neg\Box\neg\phi$; hence $(k+n) \Vdash \phi$ for some $n \geq 1$, contradiction!

Thank You!

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