Cantor's Diagonal Argument: A Characterization

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Cantor's Diagonal Argument

Its Importance

KEITH SIMMONS, The Diagonal Argument and the Liar, *Journal of Philosophical Logic* 19 (1990) 277–303.

"There are arguments found in various areas of mathematical logic that taken to form a family: the family of *diagonal arguments*. Much of recursion theory may be described as a theory of diagonalization; diagonal arguments establish basic results of set theory; and they play a central role in the proofs of limitative theorems of Gödel and Tarski. Diagonal arguments also give rise to set-theoretical and semantical paradoxes."

Cantor's Diagonal Argument

Its Ubiquity – A Part of Human Knowledge?

In Applied Mathematics:

YASUHITO TANAKA, Undecidability Of Uzawa Equivalence Theorem And Cantor's Diagonal Argument, *Applied Mathematics E-Notes* 9 (2009) 1–9.

In Economics:

ROBERT P. MURPHY, Cantor's Diagonal Argument: An Extension to the Socialist Calculation Debate, *The Quarterly J. of Australian Economics* 9:2 (2006) 3–11.

In Physics:

▷ DAVID H. WOLPERT, Physical Limits of Inference, *Physica D* 237 (2008) 1257–1281.

▷ P.-M. BINDER, Theories of Almost Everything, *Nature* 455 (2008) 884–885.

Using Cantor's Diagonalization Laplace's Demon Is Disproved...

Cantor's Diagonal Argument

Ongoing Research – A Part of Human Knowledge!

In Mathematical Logic:

SILVIO VALENTINI, Cantor Theorem and Friends, in Logical Form, *Annals of Pure and Applied Logic* 164 (2013) 502–508.

In Computer Science:

RAY WILLIAMS (IBM Almaden Research Center), Diagonalization Strikes Back: Some Recent Lower Bounds in Complexity Theory, *Proc. COCOON 2011*, LNCS 6842 (2011) 237–239.

"**Abstract.** ... In spite of its apparent weakness, the ancient method of diagonalization has played a key role in recent lower bounds. This short article ... describes a little about how diagonalization had made a recent comeback in complexity theory ...."

Cantor's Diagonal Argument

Still in Doubt?!

WILFRID HODGES, An Editor Recalls Some Hopeless Papers, *The Bulletin of Symbolic Logic* 4:1 (1998) 1–16.

"**Introduction.** I dedicate this essay to the two-dozen-odd people whose refutations of Cantor's diagonal argument ... have come to me either as a referee or an editor in the last twenty years or so. ... A few years ago it occurred to me to wonder why so many people devote so much energy to refuting this harmless little argument—what had it done to make them so angry with it?"

Cantor's Diagonal Argument

Why is it called *Diagonal*?

For $A = \{x, y, a, b, c, \cdots\}$ Put $F : A \to \mathscr{P}(A)$ as:

	x	y	a	b	c	$\cdots$	
$F(x)$	0	0	1	1	0	$\cdots$	$F(x) = \{x, y, a, b, c, \cdots\}$
$F(y)$	0	0	1	0	1	$\cdots$	$F(y) = \{x, y, a, b, c, \cdots\}$
$F(a)$	1	1	1	0	0	$\cdots$	$F(a) = \{x, y, a, b, c, \cdots\}$
$F(b)$	0	0	1	0	0	$\cdots$	$F(b) = \{x, y, a, b, c, \cdots\}$
$F(c)$	0	0	0	1	0	$\cdots$	$F(c) = \{x, y, a, b, c, \cdots\}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$

Cantor's Diagonal Argument

Why is it called *Diagonal*?

Then Diagonalize Out:

	x	y	a	b	c	$\cdots$	
$F(x)$	$\boxed{0}$	0	1	1	0	$\cdots$	$F(x)=\{x,y,a,b,c,\cdots\}$
$F(y)$	0	$\boxed{0}$	1	0	1	$\cdots$	$F(y)=\{x,y,a,b,c,\cdots\}$
$F(a)$	1	1	$\boxed{1}$	0	0	$\cdots$	$F(a)=\{x,y,a,b,c,\cdots\}$
$F(b)$	0	0	1	$\boxed{0}$	0	$\cdots$	$F(b)=\{x,y,a,b,c,\cdots\}$
$F(c)$	0	0	0	1	$\boxed{0}$	$\cdots$	$F(c)=\{x,y,a,b,c,\cdots\}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\searrow$	1	1	0	1	1		$D_F=\{x,y,a,b,c,\cdots\}$

$$D_F \neq F(x), F(y), F(a), F(b), F(c), \cdots$$

Cantor's Diagonal Argument

3rd Proof for the Uncountability of $\mathbb{R}$

JOHN FRANKS, Cantor's Other Proofs that $\mathbb{R}$ is Uncountable, *Mathematics Magazine* 83:4 (2010) 283–289. doi:10.4169/002557010X521822

Could Also Show that $A \not\approx \mathscr{P}(A)$:

For an $F : A \to \mathscr{P}(A)$ put $D_F = \{a \in A \mid a \notin F(a)\}$. Then

$$x \in D_F \longleftrightarrow x \notin F(x)$$

and so $D_F \neq F(\alpha)$ for any $\alpha \in A$:

if $D_F = F(\alpha)$ then $\quad \alpha \in D_F \longleftrightarrow \alpha \notin F(\alpha) \longleftrightarrow \alpha \notin D_F$!

Russell's Paradox

The Set of All Those
Sets Which Are Not Members of Themselves
... Does Not Exist!

Used For Showing

The Inconsistency Of The Comprehension Principle:

`For Any Formula` φ `The Set` $\{x \mid \varphi(x)\}$ `Exists.`

Because for $R = \{x \mid x \notin x\}$ we have $x \in R \longleftrightarrow x \notin x$ and so

$$R \in R \longleftrightarrow R \notin R.$$

Do We Have Another Proof For The Inconsistency Of The Comprehension Principle?

Another Formula $\psi(x)$ For Which $\{x \mid \psi(x)\}$ Is Contradictory?

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Cantor's Argument Again

For Any Surjective F The Set $\{x \mid x \notin F(x)\}$ Is Contradictory!

$D_F = \{x \mid x \notin F(x)\}$ is contradictory for surjective F:

if $D_F = F(S)$ then $S \in D_F \leftrightarrow S \notin F(S) \leftrightarrow S \notin D_F$!

For example $\bigcup$ and $\bigcap$ are surjections: $A = \bigcup\{A\} = \bigcap\{A\}$.

ZVONIMIR ŠIKIĆ, Cantor's Theorem and Paradoxical Classes, *Zeitsch. f. Math. Logik und Grundlagen d. Math.* 32 (1986) 221–226.

$$D_{\cup} = \{x \mid x \notin \bigcup x\} = \{x \mid \neg\exists y[y \in x \wedge x \in y]\} = Q_1$$

$$\{Q_1\} \in Q_1 \longleftrightarrow \nexists y\big[y \in \{Q_1\} \& \{Q_1\} \in y\big] \longleftrightarrow$$
$$\longleftrightarrow \nexists y\big[y = Q_1 \& \{Q_1\} \in y\big] \longleftrightarrow \{Q_1\} \notin Q_1.$$

Quine's Proof (inconsistency of comprehension principle)

$$Q_n = \{x \mid \neg\exists z_1, \cdots, z_n[x \in z_n \wedge \bigwedge_{i=1}^{n-1}(z_{i+1} \in z_i) \wedge z_1 \in x]\}$$

W. V. QUINE, *Mathematical Logic*, Harvard University Press (2nd ed. 1981).

$$Q_1 \in Q_1 \longrightarrow \quad \exists z_1(=Q_1)[Q_1 \in z_1 \wedge z_1 \in Q_1] \longrightarrow \quad Q_1 \notin Q_1.$$

$$Q_1 \notin Q_1 \longrightarrow \exists z_1[Q_1 \in z_1 \wedge z_1 \in Q_1] \longrightarrow$$

$$[z_1 \in Q_1 \wedge Q_1 \in z_1] \longrightarrow$$

$$\exists u_1(=Q_1)[z_1 \in u_1 \wedge u_1 \in z_1] \longrightarrow$$

$z_1 \notin Q_1 \longrightarrow$ ⋇ contradiction !

Šikić's Proof of Quine

$$Q_n = \{x \mid \neg\exists z_1, \cdots, z_n[x \in z_n \wedge \bigwedge_{i=1}^{n-1}(z_{i+1} \in z_i) \wedge z_1 \in x]\}$$

$$Q_n = \{x \mid x \notin \bigcup^n x\}$$

$$\{\!\{Q_n\}\!\}^n \notin Q_n \longleftrightarrow \{\!\{Q_n\}\!\}^n \in \bigcup^n \{\!\{Q_n\}\!\}^n \longleftrightarrow$$
$$\exists z_1, \cdots, z_n[\{\!\{Q_n\}\!\}^n \in z_n \wedge \bigwedge_{i=1}^{n-1}(z_{i+1} \in z_i) \wedge z_1 \in \{\!\{Q_n\}\!\}^n]$$
$$\longleftrightarrow \exists z_1, \cdots, z_n[\{\!\{Q_n\}\!\}^n \in z_n \wedge \bigwedge_{j=1}^{n} z_j = \{\!\{Q_n\}\!\}^{n-j}]$$
$$\longleftrightarrow \exists z_n[\{\!\{Q_n\}\!\}^n \in z_n \wedge z_n = Q_n] \longleftrightarrow \{\!\{Q_n\}\!\}^n \in Q_n.$$

More Russell–Like Paradox

The Set of All The Things Of Sets
Which Do Not Belong To The Sets
... Does Not Exist!

For example, the set of all singletons which do not belong to their (only) member $\{\{y\}\mid\{y\}\notin y\}$ does not exist:

since if $O=\{x\mid\exists y\big[\underbrace{\forall z(z\in x\leftrightarrow z=y)}_{x=\{y\}}\wedge x\notin y\big]\}$ then

$$\{O\}\in O\longleftrightarrow\exists y[\{O\}=\{y\}\wedge\{O\}\notin y]\longleftrightarrow\{O\}\notin O.$$

Many More Examples:

$T=\big\{\{\emptyset,y\}\mid y\neq\emptyset\wedge\{\emptyset,y\}\notin y\big\}$ $\quad P=\{\mathscr{P}(y)\mid\mathscr{P}(y)\notin y\}$

$F=\{y\times y\mid y\times y\notin y\}$, etc.

A General Pattern:

$$\{G(y) \mid G(y) \notin y\} = \{x \mid \exists y[x = G(y) \,\&\, x \notin y]\}$$

If $R_G = \{G(y) \mid G(y) \notin y\}$ Then

- $G(R_G) \notin R_G \longrightarrow G(R_G) \in R_G$, Contradiction!
- $G(R_G) \in R_G \longrightarrow G(R_G) = G(y) \notin y \rightarrow_{G \text{ injective}} \rightarrow R_G = y$
  $\longrightarrow G(R_G) \notin R_G$, Contradiction!

So, lots of proofs for the inconsistency of $\{x \mid \varphi(x)\}$ by surjective F's ($D_F = \{x \mid x \notin F(x)\}$) and injective G's ($R_G = \{x \mid \exists y[x = G(y) \wedge x \notin y]\}$).

Note that every surjection has an injective right inverse, and every injection has a surjective left inverse.

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The Paradox of Well–Founded Sets

The Set of All Those
Sets Whose Every Membership Chain Finitely Terminates
Does Not Exist!

$$Q_\infty = \{x \mid \neg\exists z_1, z_2, \cdots [\bigwedge_{i=1}^{\infty}(z_{i+1} \in z_i) \wedge z_1 \in x]\}$$

- $Q_\infty \in Q_\infty \longrightarrow \cdots \in Q_\infty \in Q_\infty \in Q_\infty \in Q_\infty$
  $\longrightarrow \exists z_1, z_2, \cdots [\bigwedge_{i=1}^{\infty}(z_{i+1} \in z_i) \wedge z_1 \in Q_\infty]$
  $\longrightarrow Q_\infty \notin Q_\infty$
- $Q_\infty \notin Q_\infty \longrightarrow \exists z_1, z_2, \cdots [\bigwedge_{i=1}^{\infty}(z_{i+1} \in z_i) \wedge z_1 \in Q_\infty]$
  $\longrightarrow \exists z_1 \Big(\exists z_2, \cdots [\bigwedge_{i=2}^{\infty}(z_{i+1} \in z_i) \wedge z_2 \in z_1] \wedge z_1 \in Q_\infty\Big)$
  $\longrightarrow \exists z_1 \big(z_1 \notin Q_\infty \wedge z_1 \in Q_\infty\big) \longrightarrow$ ⋇ contradiction !

The Paradox of Well–Founded Sets

The Set of All Those
Sets Whose Every Membership Chain Finitely Terminates
Does Not Exist!

SHEN YUTING, Paradox of the Class of All Grounded Classes, *The Journal of Symbolic Logic* 18:2 (1953) 114.

$$Q_\infty = \{x \mid \neg\exists z_1, z_2, \cdots [\bigwedge\!\!\!\bigwedge_{i=1}^{\infty}(z_{i+1} \in z_i) \wedge z_1 \in x]\}$$

The function $G(x) = \begin{cases} x & \text{if } x \in Q_\infty \\ \{x\} & \text{if } x \notin Q_\infty \end{cases}$ is injective, since $x \in Q_\infty \longleftrightarrow \{x\} \in Q_\infty$, and we have

$$R_G = \{G(x) \mid G(x) \notin x\} = Q_\infty$$

Cantor's Diagonal Argument: Some Variants

Permuting Rows or Columns

For $A=\{x,y,a,b,c,\cdots\}$ and $F:A\to\mathscr{P}(A)$

	x	y	a	b	c	$\cdots$	
$F(x)$	0	0	1	1	0	$\cdots$	$F(x)=\{x,y,a,b,c,\cdots\}$
$F(y)$	0	0	1	0	1	$\cdots$	$F(y)=\{x,y,a,b,c,\cdots\}$
$F(a)$	1	1	1	0	0	$\cdots$	$F(a)=\{x,y,a,b,c,\cdots\}$
$F(b)$	0	0	1	0	0	$\cdots$	$F(b)=\{x,y,a,b,c,\cdots\}$
$F(c)$	0	0	0	1	0	$\cdots$	$F(c)=\{x,y,a,b,c,\cdots\}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$

Cantor's Diagonal Argument: Some Variants

Permuting Rows

Permute rows by $h : A \to A$ as

$h(x) = a, \ h(y) = b, \ h(a) = y, \ h(b) = c, \ h(c) = x$

	x	y	a	b	c	$\cdots$	
$F(h(x))$	1	1	1	0	0	$\cdots$	$F(a) = \{x, y, a, b, c, \cdots\}$
$F(h(y))$	0	0	1	0	0	$\cdots$	$F(b) = \{x, y, a, b, c, \cdots\}$
$F(h(a))$	0	0	1	0	1	$\cdots$	$F(y) = \{x, y, a, b, c, \cdots\}$
$F(h(b))$	0	0	0	1	0	$\cdots$	$F(c) = \{x, y, a, b, c, \cdots\}$
$F(h(c))$	0	0	1	1	0	$\cdots$	$F(x) = \{x, y, a, b, c, \cdots\}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$

Cantor's Diagonal Argument: Some Variants

Permuting Rows

Diagonalizing Out:

	x	y	a	b	c	$\cdots$	
$F(h(x))$	$\boxed{1}$	1	1	0	0	$\cdots$	$F(a)=\{x,y,a,b,c,\cdots\}$
$F(h(y))$	0	$\boxed{0}$	1	0	0	$\cdots$	$F(b)=\{x,y,a,b,c,\cdots\}$
$F(h(a))$	0	0	$\boxed{1}$	0	1	$\cdots$	$F(y)=\{x,y,a,b,c,\cdots\}$
$F(h(b))$	0	0	0	$\boxed{1}$	0	$\cdots$	$F(c)=\{x,y,a,b,c,\cdots\}$
$F(h(c))$	0	0	1	1	$\boxed{0}$	$\cdots$	$F(x)=\{x,y,a,b,c,\cdots\}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\searrow$	0	1	0	0	1		$D_{F\circ h}=\{x,y,a,b,c,\cdots\}$

Cantor's Diagonal Argument: Some Variants

Permuting Rows

For any $F : A \to \mathscr{P}(A)$ and any surjection $h : A \to A$, put

$$D_{F\circ h} = \{a \in A \mid a \notin F(h(a))\}$$

If $D_{F\circ h} = F(\alpha)$ and $h(\beta) = \alpha$ (by surjectivity of h), then

$$\beta \in D_{F\circ h} \longleftrightarrow \beta \notin F(h(\beta)) \longleftrightarrow \beta \notin F(\alpha) \longleftrightarrow \beta \notin D_{F\circ h}$$

Is Any Set ($B \subseteq A$) Not In The Range Of F ($B \neq F(\square)$) In The Form Of $D_{F\circ h}$ For Some (SURJECTION) h?

ROBERT GRAY, George Cantor and Transcendental Numbers, *The American Mathematical Monthly* 101:9 (1994) 819–832.

Theorem: A real number in the interval $(0,1)$ is transcendental if and only if it is the diagonal number of a sequence that consists of all the binary representations of algebraic reals in $(0,1)$.

Cantor's Diagonal Argument: Some Variants

Permuting Rows

For No Surjective h Can We Have $D_{F\circ h}=\{b,c\}$:

	x	y	a	b	c	
$F(x)$	0	0	1	1	0	$F(x)=\{x,y,a,b,c\}$
$F(y)$	0	0	1	0	1	$F(y)=\{x,y,a,b,c\}$
$F(a)$	1	1	1	0	0	$F(a)=\{x,y,a,b,c\}$
$F(b)$	0	0	1	0	0	$F(b)=\{x,y,a,b,c\}$
$F(c)$	0	0	0	1	0	$F(c)=\{x,y,a,b,c\}$

$$D_{F\circ h}=\{a\in A \mid a\notin F(h(a))\}$$

$x\notin D_{F\circ h}\longrightarrow x\in F(h(x))\longrightarrow h(x)=a$

$y\notin D_{F\circ h}\longrightarrow y\in F(h(y))\longrightarrow h(y)=a$

So h cannot be injective and (by A's finiteness) cannot be surjective.

Cantor's Diagonal Argument: Some Variants

Permuting Columns

Permute columns by $g : A \to A$ as

$$g(x) = x,\ \ g(y) = a,\ \ g(a) = c,\ \ g(b) = y,\ \ g(c) = b$$

	x	a	c	y	b	$\cdots$	
$F(x)$	0	1	0	0	1	$\cdots$	$F(x) = \{x, a, c, y, b, \cdots\}$
$F(y)$	0	1	1	0	0	$\cdots$	$F(y) = \{x, a, c, y, b, \cdots\}$
$F(a)$	1	1	0	1	0	$\cdots$	$F(a) = \{x, a, c, y, b, \cdots\}$
$F(b)$	0	1	0	0	0	$\cdots$	$F(b) = \{x, a, c, y, b, \cdots\}$
$F(c)$	0	0	0	0	1	$\cdots$	$F(c) = \{x, a, c, y, b, \cdots\}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$

Cantor's Diagonal Argument: Some Variants

Permuting Columns

Diagonalizing Out:

	x	a	c	y	b	$\cdots$	
$F(x)$	$\boxed{0}$	1	0	0	1	$\cdots$	$F(x)=\{x,a,c,y,b,\cdots\}$
$F(y)$	0	$\boxed{1}$	1	0	0	$\cdots$	$F(y)=\{x,a,c,y,b,\cdots\}$
$F(a)$	1	1	$\boxed{0}$	1	0	$\cdots$	$F(a)=\{x,a,c,y,b,\cdots\}$
$F(b)$	0	1	0	$\boxed{0}$	0	$\cdots$	$F(b)=\{x,a,c,y,b,\cdots\}$
$F(c)$	0	0	0	0	$\boxed{1}$	$\cdots$	$F(c)=\{x,a,c,y,b,\cdots\}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\searrow$	1	0	1	1	0		$D_F^g=\{x,a,c,y,b,\cdots\}$

Cantor's Diagonal Argument: Some Variants

Permuting Columns

For any $F : A \to \mathscr{P}(A)$ and any injection $g : A \to A$, put

$$D_F^g = \{g(a) \in A \mid g(a) \notin F(a)\}$$

If $D_F^g = F(\alpha)$, then

- $g(\alpha) \notin F(\alpha) \longrightarrow g(\alpha) \in D_F^g \longrightarrow g(\alpha) \in F(\alpha)$.
- $g(\alpha) \in F(\alpha) \longrightarrow \exists x[g(\alpha) = g(x) \notin F(x)] \longrightarrow g(\alpha) \notin F(\alpha)$.

Is Any Set ($B \subseteq A$) Not In The Range Of F ($B \neq F(\square)$) In The Form Of D_F^g For Some (INJECTION) g?

Cantor's Diagonal Argument: Some Variants

Permuting Columns

For No Injective g Can We Have $D_F^g = \{b, c\}$:

	x	y	a	b	c	
$F(x)$	0	0	1	1	0	$F(x) = \{x, y, a, b, c\}$
$F(y)$	0	0	1	0	1	$F(y) = \{x, y, a, b, c\}$
$F(a)$	1	1	1	0	0	$F(a) = \{x, y, a, b, c\}$
$F(b)$	0	0	1	0	0	$F(b) = \{x, y, a, b, c\}$
$F(c)$	0	0	0	1	0	$F(c) = \{x, y, a, b, c\}$

$$D_F^g = \{g(a) \in A \mid g(a) \notin F(a)\}$$

$x \notin D_F^g \longrightarrow_g$ surjective $x = g(\square) \in F(\square) \longrightarrow \square = a \longrightarrow g(a) = x$

$y \notin D_F^g \longrightarrow_g$ surjective $y = g(\square) \in F(\square) \longrightarrow \square = a \longrightarrow g(a) = y$

So, no (injective / surjective) function g can satisfy $D_F^g = \{b, c\}$!

Cantor's Diagonal Argument: Some Variants

$n-$Circularity and $\infty-$Circularity à la Quine

▷ NATARAJAN RAJA, A Negation-Free Proof of Cantor's Theorem, *Notre Dame Journal of Formal Logic* 46:2 (2005) 231–233.

▷ NATARAJAN RAJA, Yet Another Proof of Cantor's Theorem, *Dimensions of Logical Concepts*, Coleção CLE, Vol. 54 (2009) 209–217.

For $F : A \to \mathscr{P}(A)$ Let

$$D_F^{\infty} = \{x \in A \mid \neg\exists z_1, z_2, \cdots [\bigwedge_{i=1}^{\infty}(z_{i+1} \in F(z_i)) \wedge z_1 \in F(x)]\}$$

$$D_F^{n} = \{x \mid \neg\exists z_1, z_2, \cdots, z_n[x \in F(z_n) \wedge \bigwedge_{i=1}^{\infty}(z_{i+1} \in F(z_i)) \wedge z_1 \in F(x)]\}$$

One can show that none of these can be in the range of F:

$$D_F^{\infty} \subseteq \cdots \subseteq \{D_F^n\}_n \subseteq \cdots \subseteq D_F$$

Cantor's Diagonal Argument: Some Variants

$n-$Circularity and $\infty-$Circularity à la Quine

$$[M_{n\times n} \boxtimes N_{n\times n}]_{i,j} = \bigvee_k (M_{i,k} \wedge N_{k,j})$$

For $A = \{x, y, a, b, c\}$ and $F : A \to \mathscr{P}(A)$

	x	y	a	b	c		x	y	a	b	c
$F(x)$	0	0	1	1	0		0	0	1	1	0
$F(y)$	0	0	1	0	1		0	0	1	0	1
$F(a)$	1	1	1	0	0		1	1	1	0	0
$F(b)$	0	0	1	0	0		0	0	1	0	0
$F(c)$	0	0	0	1	0		0	0	0	1	0

$$[M \boxtimes M]_{i,j} = 1 \iff \exists k[k \in F(i) \wedge j \in F(k)]$$

Cantor's Diagonal Argument: Some Variants

$n-$Circularity and $\infty-$Circularity à la Quine

Let $F^2(i) = \{j \mid \exists k[j \in F(k) \wedge k \in F(i)]\}$

	x	y	a	b	c	
$F^2(x)$	$\boxed{1}$	1	1	0	0	$F^2(x) = \{x, y, a, b, c\}$
$F^2(y)$	1	$\boxed{0}$	1	1	0	$F^2(y) = \{x, y, a, b, c\}$
$F^2(a)$	1	1	$\boxed{1}$	1	1	$F^2(a) = \{x, y, a, b, c\}$
$F^2(b)$	1	1	1	$\boxed{0}$	0	$F^2(b) = \{x, y, a, b, c\}$
$F^2(c)$	0	0	1	0	$\boxed{0}$	$F^2(c) = \{x, y, a, b, c\}$
$\searrow$	0	0	0	1	1	$D_F^2 = \{x, y, a, b, c\}$

Cantor's Diagonal Argument: A Characterization

$$D_F^g = \{g(x) \in A \mid g(x) \notin F(x)\}$$

If $D_F^g = F(\alpha)$ Then

- $g(\alpha) \notin D_F^g \longrightarrow g(\alpha) \in F(\alpha) \longrightarrow g(\alpha) \in D_F^g$!
- if g is injective w.r.t F, i.e., $g(x) = g(y) \Longrightarrow F(x) = F(y)$:

$$g(\alpha) \in D_F^g \longrightarrow \exists x : g(\alpha) = g(x) \notin F(x) \longrightarrow$$
$$F(\alpha) = F(x) \longrightarrow g(\alpha) \notin F(\alpha) \longrightarrow g(\alpha) \notin D_F^g!$$

- if g is $F-$injective, i.e., $g(x) = g(y) \Longrightarrow \{g(x)\} \cap F(x) = \{g(y)\} \cap F(y)$:

$$g(\alpha) \in D_F^g \longrightarrow \exists x : g(\alpha) = g(x) \notin F(x) \longrightarrow$$
$$\{g(\alpha)\} \cap F(\alpha) = \{g(x)\} \cap F(x) = \emptyset \longrightarrow$$
$$g(\alpha) \notin F(\alpha) \longrightarrow g(\alpha) \notin D_F^g!$$

Cantor's Diagonal Argument: A Characterization

$$D_F^g = \{g(x) \in A \mid g(x) \notin F(x)\}$$

Any Subset B of A is Not in the Range of $F : A \to \mathscr{P}(A)$ (i.e., $B \neq F(a)$ for all $a \in A$) If And Only If There Exists Some $F-$injective $g : A \to A$ Such That $B \cap g[A] = D_F^g$.

Proof.

If $B \neq F(a)$ then there is some $b \in (B - F(a)) \cup (F(a) - B)$; call it $g(a)$. Then g is $F-$injective, since if $g(x) = g(y)$ then $g(x) \in F(x) \leftrightarrow g(x) \notin B \leftrightarrow g(y) \notin B \leftrightarrow g(y) \in F(y)$.
Now, from $g(x) \in B \leftrightarrow g(x) \notin F(x)$ we have $B \cap g[A] = D_F^g$. □

Cantor's Diagonal Argument: A Characterization

Characterizing $D_F^{\propto}$ for $\propto = 1, 2, \cdots, \infty$

Put

$$g(x) = \begin{cases} x & \text{if } x \in D_F^{\propto} \\ z_1 \, [\in F(x)] & \text{if } x \notin D_F^{\propto} \big(\exists z_{\propto}, \cdots, z_1 [\cdots \wedge z_1 \in F(x)]\big) \end{cases}$$

Then g is $F-$injective and $D_F^{\propto} = D_F^{g}$:

$$g(x) = x \longleftrightarrow x \in D_F^{\propto} \longleftrightarrow g(x) \notin F(x)$$
$$g(x) \neq x \longleftrightarrow x \notin D_F^{\propto} \longleftrightarrow g(x) \in F(x)$$

If $g(x) = g(y)$ Then $x \in D_F^{\propto} \longleftrightarrow y \in D_F^{\propto}$ and $g(x) \in F(x) \longleftrightarrow g(y) \in F(y)$ so $\{g(x)\} \cap F(x) = \{g(y)\} \cap F(y)$.

Cantor's Diagonal Argument

In Recursion Theory (and Computer Science)

ENRIQUE ALONSO & MARIA MANZANO, Diagonalisation and Church's Thesis: Kleene's Homework, *History and Philosophy of Logic* 26 (2005) 93–113.

Kleene (1981):

> "When Church proposed this thesis, I sat down to disprove it by diagonalizing out of the class of the $\lambda-$definable functions. But, quickly realizing that the diagonalization cannot be done effectively, I became overnight a supporter of the thesis."

Let the $n^{\rm th}$ Computable (Recursive) (unary) Function be φ_n and denote its domain by $\mathscr{W}_n$; i.e., $\mathscr{W}_n = \{x \in \mathbb{N} \mid \exists y[\varphi_n(x) = y]\}$ is the $n^{\rm th}$ Recursively Enumerable (RE) set.

Cantor's Diagonal Argument In Computability Theory

Turing's Halting Problem

Diagonalizing Out of the sequence $\mathscr{W}_0, \mathscr{W}_1, \mathscr{W}_2, \cdots$ of all the RE sets, we get the non–RE set $\overline{K} = \{x \in \mathbb{N} \mid x \notin \mathscr{W}_x\}$.

By accident, its complement $K = \{x \mid x \in \mathscr{W}_x\}$ is RE.

So, we have an undecidable set K and an algorithmically unsolvable problem:

Turing's Halting Problem: for any algorithm and any of its inputs, determine whether or not that algorithm eventually halts (after running) on that input.

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Cantor's Diagonal Argument In Computability Theory

(Effectively) Non–RE Sets

By our characterization we know that any non–RE set $P \subseteq \mathbb{N}$ satisfies $P \cap g[\mathbb{N}] = \{g(n) \mid g(n) \notin \mathscr{W}_n\}$ for some $\mathscr{W}$–injection g (i.e., $g(x) = g(y) \longrightarrow \{g(x)\} \cap \mathscr{W}_x = \{g(y)\} \cap \mathscr{W}_y$). For g we have

$$g(n) \in P \longleftrightarrow g(n) \notin \mathscr{W}_n$$

P is called *Completely Productive* (EFFECTIVELY NON–RE) if for some *computable* g : $g(x) \in (P - \mathscr{W}_x) \cup (\mathscr{W}_x - P)$

RAYMOND M. SMULLYAN, *Recursion Theory for Metamathematics*, Oxford University Press (1993).

There are some non–RE sets which are not effectively non–RE (completely productive).

Cantor's Diagonal Argument In Computability Theory

(Effectively) Non–RE Sets

P is called *Productive* if for some computable f (and any x)

$$\mathscr{W}_x \subseteq P \longrightarrow f(x) \in P - \mathscr{W}_x$$

P = The Set of True Arithmetical Formulas
$\mathscr{W}_x$ = An RE Sound Theory
$f(x)$ = A True but Unprovable Formula

(Effective) Gödel's First Incompleteness Theorem:

For Any Sound and RE Arithmetical Theory (which is sufficiently expressive and strong) There Exists Some Arithmetical Sentence Which is True and Unprovable in the Theory.

Cantor's Diagonal Argument In Computability Theory

(Effectively) Non–RE Sets

P is called *Productive* if for some computable f: $\forall x$

$$\mathscr{W}_x \subseteq P \longrightarrow f(x) \in P - \mathscr{W}_x$$

P is called *Completely Productive* if for some *computable* g: $\forall x$

$$g(x) \in (P - \mathscr{W}_x) \cup (\mathscr{W}_x - P)$$

JOHN MYHILL, Creative Sets, *Zeitsch. f. Math. Logik und Grundlagen d. Math.* 1 (1955) 97–108. P is Productive $\iff$ P is Completely Productive

$\iff$ $P \cap g[\mathbb{N}] = \{g(n) \mid g(n) \notin \mathscr{W}_n\}$ for some regular g

J. C. E. DEKKER, Productive Sets, *Transactions of the American Mathematical Society* 78 (1955) 129–149.

Cantor's Diagonal Argument In Computability Theory

(Completely) Productive and Creative Sets

PIERGIORGIO ODIFREDDI, *Classical Recursion Theory: The Theory of Functions and Sets of Natural Numbers – Vol. 1*, North–Holland (1989).

$C \subseteq \mathbb{N}$ is *Creative* when C is RE and $C^{\complement} = \mathbb{N} - C$ is productive

Post (1944):

> "every symbolic logic is incomplete and extendible relative to the class of propositions constituting $[K]$. The conclusion is inescapable that even for such a fixed, well defined body of mathematical propositions, *mathematical thinking is, and must remain, essentially creative.*"

Cantor's Diagonal Argument In Computability Theory

(Completely) Productive and Creative Sets

BRUCE M. HOROWITZ, Sets Completely Creative via Recursive Permutations, *Zeitsch. f. Math. Logik und Grundlagen d. Math.* 24 (1978) 445–452.

C is creative if and only if $C = \{\pi(x) \mid \pi(x) \in \mathscr{W}_x\}$ for some recursive permutation π

This in effect says that

a set is effectively non–RE (productive = completely productive) and has an RE complement if and only if is of the form $D^g_{\mathscr{W}} = \{g(x) \mid g(x) \notin \mathscr{W}_x\}$ for some computable permutation g.

Cantor's Diagonal Argument: A Characterization

Characterizing Diagonally Proved Sets

ZVONIMIR ŠIKIĆ, The Diagonal Argument—A Study of Cases, *International Studies in the Philosophy of Science* 6:3 (1992) 191–203.

Many More Sets Have Been Proved To Exist By The Diagonal Argument; Almost All of Them Can Be Characterized.

An Ongoing (Long) Project: Characterizing The Diagonally Proved Sets (in any field) And Getting More Instances of Those Sets ...

Thank You!

Thanks to The Participants
for Listening and for Their Patience!
and Thanks to The Organizers
For Everything!

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