Gödel's Incompleteness from a Computational Viewpoint

Saeed Salehi

University of Tabriz

http://SaeedSalehi.ir/

Completeness vs. Incompleteness of Kurt Gödel

Completeness of Logic $\mathfrak{L}$ with respect to Class of Structures $\mathscr{K}$:

For any formula φ: $\forall\mathcal{M}\in\mathscr{K}\,(\mathcal{M}\models\varphi)\implies\;\vdash_{\mathfrak{L}}\varphi$.

Strong Completeness

For any theory Γ (set of formulas) and any formula φ:

$$\forall\mathcal{M}\in\mathscr{K}\,(\mathcal{M}\models\Gamma\Rightarrow\mathcal{M}\models\varphi)\implies\Gamma\vdash_{\mathfrak{L}}\varphi.$$

Soundness of Logic $\mathfrak{L}$ with respect to Class of Structures $\mathscr{K}$:

For any formula φ: $\vdash_{\mathfrak{L}}\varphi\implies\forall\mathcal{M}\in\mathscr{K}\,(\mathcal{M}\models\varphi)$.

$\equiv$ Strong Soundness

$\forall\Gamma\,\forall\varphi$: $\Gamma\vdash_{\mathfrak{L}}\varphi\Longrightarrow\forall\mathcal{M}\in\mathscr{K}\,(\mathcal{M}\models\Gamma\Rightarrow\mathcal{M}\models\varphi)$.

So (*here*) Completeness & Soundness are Semantic concepts.

Completeness vs. Incompleteness of Kurt Gödel

Completeness of Theory T w.r.t Class of Structures $\mathscr{K}$:

For any formula φ: $\forall\mathcal{M}\in\mathscr{K}(\mathcal{M}\models\mathrm{T}\Rightarrow\mathcal{M}\models\varphi)\Longrightarrow\mathrm{T}\vdash_{\mathfrak{L}}\varphi$.

Soundness of Theory T w.r.t Class of Structures $\mathscr{K}$:
For any formula φ: $\mathrm{T}\vdash_{\mathfrak{L}}\varphi\Longrightarrow\forall\mathcal{M}\in\mathscr{K}(\mathcal{M}\models\mathrm{T}\Rightarrow\mathcal{M}\models\varphi)$.

The Theory T axiomatizes the Class $\mathscr{K}$:
T is Sound and Complete w.r.t $\mathscr{K}$; $\mathrm{T}=\mathtt{AxTh}(\mathscr{K})$; $\mathscr{K}=\mathtt{Mod}(\mathrm{T})$.

(SEMANTIC) $\mathscr{K}$ is axiomatizable iff $\mathscr{K}=\mathtt{Mod}(\mathtt{Th}(\mathscr{K}))$ iff
$\mathscr{K}$ is closed under elementary equivalence and ultra-products
iff $\mathscr{K}$ is an elementary class.

(SYNTACTIC) $\mathtt{Der}(\mathrm{T})=\{\theta\mid\mathrm{T}\vdash\theta\}=\mathtt{Th}(\mathtt{Mod}(\mathrm{T}))$.

Completeness vs. Incompleteness of Kurt Gödel

Syntactic Completeness of Theory T:
For any formula φ: either $\mathrm{T}\vdash_{\mathfrak{L}}\varphi$ or $\mathrm{T}\vdash_{\mathfrak{L}}\neg\varphi$.

That is *Negation* Completeness: $\mathrm{T}\vdash_{\mathfrak{L}}\neg\varphi \Longleftarrow \mathrm{T}\nvdash_{\mathfrak{L}}\varphi$.
Conjunction Completeness: $\mathrm{T}\vdash_{\mathfrak{L}}\varphi\wedge\psi \iff \mathrm{T}\vdash_{\mathfrak{L}}\varphi \;\&\; \mathrm{T}\vdash_{\mathfrak{L}}\psi$.
Disjunction Completeness: $\mathrm{T}\vdash_{\mathfrak{L}}\varphi\vee\psi \iff \mathrm{T}\vdash_{\mathfrak{L}}\varphi \text{ or } \mathrm{T}\vdash_{\mathfrak{L}}\psi$.

It all makes sense in the case of
Consistency of Theory T:
For any formula φ: either $\mathrm{T}\nvdash_{\mathfrak{L}}\varphi$ or $\mathrm{T}\nvdash_{\mathfrak{L}}\neg\varphi$.

$$\mathrm{T}\vdash_{\mathfrak{L}}\neg\varphi \Longrightarrow \mathrm{T}\nvdash_{\mathfrak{L}}\varphi.$$

Completeness vs. Incompleteness of Kurt Gödel

(SYNTACTIC) Completeness and Consistency $\equiv$
(SEMANTIC) Completeness and Soundness w.r.t a Class of
Equivalent Models.

$\equiv \forall\varphi: \quad T \vdash \neg\varphi \iff T \nvdash \varphi.$

(Syntactic) Complete + Consistent $\iff$ Maximally Consistent.

So, by Axiom of Choice, every Theory *can be* COMPLETED.

`But not in an effective (algorithmic) way !`

Completeness vs. Incompleteness of Kurt Gödel

Axiomatizable Theory: A Consistent Theory whose Axioms can be Algorithmically Listed (be Recursively Enumerable).
Then, the Theorems of the Theory will be R.E. too.

A(n Axiomatizable) Theory is called *Decidable* if the set of its Theorems is Decidable (Recursive).
A(n Axiomatizable) Theory T is *Completable* if there exists a(n axiomatizable) Complete Theory T' extending T, i.e., $\mathrm{T} \subseteq \mathrm{T}'$.

From a Logician's Point of View:

(SYNTACTIC) Complete $\Longrightarrow$ Decidable $\Longrightarrow$ Completable.

Completeness vs. Incompleteness of Kurt Gödel

T is Complete $\Longrightarrow$ T is Decidable:

Since $\{\theta \mid \mathrm{T} \vdash \theta\}$ is R.E. then $\{\theta \mid \mathrm{T} \nvdash \theta\} = \{\theta \mid \mathrm{T} \vdash \neg\theta\}$ is R.E.
So, $\{\theta \mid \mathrm{T} \vdash \theta\}$ is Decidable (Recursive).

T is Decidable $\Longrightarrow$ T is Completable:

The Henkin Construction for a Completion of T is `effective`,
for T is Decidable.

Thus that Completion is also Decidable; so T is Completable.

Completeness vs. Incompleteness of Kurt Gödel

Does Decidability (of T) $\Longrightarrow$ Completeness (of T)?

NO: Monadic Predicate Logic (without Equality –
Unary Relations Only [like $P(x)$]).
Decidable but Incomplete ($\nvdash \forall x P(x)$ & $\nvdash \exists x \neg P(x)$).

$$\text{Completeness} \Longrightarrow \text{Decidability}.$$
$$\not\Longleftarrow$$

Completeness vs. Incompleteness of Kurt Gödel

Does Completability (of T) $\Longrightarrow$ Decidability (of T)?

NO: First-Order Logic with equality is UNDecidable,
but Completable:

$$\text{Logic} + \forall x \forall y (x = y).$$

$$\text{Decidability} \underset{\not\Longleftarrow}{\Longrightarrow} \text{Completability}.$$

Completeness vs. Incompleteness of Kurt Gödel

$$\text{Incompletability} \Longrightarrow \text{Undecidability} \Longrightarrow \text{Incompleteness}$$
$$\not\Longleftarrow \qquad\qquad\qquad \not\Longleftarrow$$

Incompletable = Essentially Undecidable

A Simple Example of an Incompletable Theory ?
With a Simple Proof of its Incompletability?

Gödel's Incompleteness Theorem ...

Kurt Gödel's Incompleteness . . . COMPUTATIONALLY

A Complete Theory

Axioms A_L over the language $\langle 0, \mathbf{S}, < \rangle$:

- $\forall x \forall y (x < y \rightarrow y \not< x)$
- $\forall x \forall y \forall z (x < y \wedge y < z \rightarrow x < z)$
- $\forall x \forall y (x < y \vee x = y \vee y < x)$
- $\forall x (x \not< 0)$
- $\forall x \forall y (x < \mathbf{S}(y) \leftrightarrow x < y \vee x = y)$
- $\forall x (x \neq 0 \rightarrow \exists y [y = \mathbf{S}(x)])$

This Axiomatizes the Theory $\langle \mathbb{N}, 0, \mathbf{S}, < \rangle$.

Kurt Gödel’s Incompleteness . . . COMPUTATIONALLY

A Ternary Predicate $\mathcal{T}(e,x,t)$=
The (single-input) Algorithm (with code) e with input x takes time t to halt (and it indeed halt).

Let the Theory A_S be A_L +

$$\{\mathcal{T}(\overline{e},\overline{x},\overline{t}) \mid \mathbb{N} \models \mathcal{T}(e,x,t)\}$$

where $\overline{n}$ is $\underbrace{\mathbf{S}\cdots\mathbf{S}}_{n \text{ times}}(0)$.

Theory A_S is UnDecidable but Completable.

Kurt Gödel's Incompleteness . . . COMPUTATIONALLY

A Completion:

$$A_S + \forall y \forall x \forall z \mathcal{T}(y, x, z).$$

UnDecidability of A_S:
Was A_S decidable then Halting Problem would be solvable:
Take e and x, form $\varphi_{e,x} \; = \; \exists t \mathcal{T}(\overline{e}, \overline{x}, z)$.
$A_S \vdash \varphi_{e,x} \iff \mathbb{N} \models \mathcal{T}(e, x, t)$ for some $t \in \mathbb{N} \iff$
Program e with Input x eventually halts.

Kurt Gödel's Incompleteness . . . COMPUTATIONALLY

UnDecidability of A_S Directly:

If $\{\theta \mid A_S \vdash \theta\}$ is Decidable, then so is

$$\mathfrak{D} = \{n \mid A_S \nvdash \exists z \mathcal{T}(\overline{n}, \overline{n}, z)\}.$$

Let the Algorithm (with code) e halt on x whenever $x \in \mathfrak{D}$ and does not halt (loop forever) whenever $x \notin \mathfrak{D}$.

Then Algorithm (with code) e with input e:

- (Algorithm e Halts in time t on input e) $\iff$
  $\iff [\mathbb{N} \models \mathcal{T}(n, n, t)] \iff [\mathcal{T}(\overline{n}, \overline{n}, \overline{t}) \in A_S] \iff$
  $\iff [A_S \vdash \exists z \mathcal{T}(\overline{e}, \overline{e}, z)] \iff [e \notin \mathfrak{D}] \iff$
  (Algorithm e does NOT halt on input e)!

The Proof Works for Every Sound $T \supseteq A_S$ (s.t. $\mathbb{N} \models T$).

So, A_S is NOT Soundly Completable.

Kurt Gödel’s Incompleteness . . . COMPUTATIONALLY

So, we can complete A_S as $A_S + \forall y \forall x \forall z \mathcal{T}(y, x, z)$.
But there is no complete $T \supseteq A_S$ such that $\mathbb{N} \models T$.
Thus $\mathtt{Th}(\mathbb{N}, 0, \mathbf{S}, <, \mathcal{T})$ is NOT R.E.

The Proof is the Classical Argument:
A Sound Theory (of $\mathbb{N}$) Can Not Be Complete: Because of the Existence of a Definable non-E.R. Set, or an R.E. Set Which is Not Decidable. For example, $K = \{n \in \mathbb{N} \mid n \in W_n\}$ is R.E. and UnDecidable. Thus $\overline{K} = \{n \mid n \notin W_n\}$ is not R.E. For a Sound Theory T, we have the R.E. Set $\{m \mid T \vdash \text{“}m \notin W_m\text{”}\} \subset \overline{K}$. So, there must Exist some $n \in \overline{K}$ for which $T \nvdash \text{“}n \notin W_n\text{”}$. Thus $(\mathbb{N} \models)\text{“}n \notin W_n\text{”}$ is a True Sentence which is Not $T-$Provable.

Kurt Gödel's Incompleteness . . . COMPUTATIONALLY

Let A_T be A_S + $\{\neg\mathcal{T}(\overline{e},\overline{x},\overline{t}) \mid \mathbb{N} \models \neg\mathcal{T}(e,x,t)\}$
in a Language that Contains a (Definable) Pairing Function π.

So, A_T is Axiomatized over $\langle 0, \mathbf{S}, <, \mathcal{T}, \pi\rangle$ by

- $\forall x \forall y (x < y \rightarrow y \not< x)$
- $\forall x \forall y \forall z (x < y \wedge y < z \rightarrow x < z)$
- $\forall x \forall y (x < y \vee x = y \vee y < x)$
- $\forall x (x \not< 0)$
- $\forall x \forall y (x < \mathbf{S}(y) \leftrightarrow x < y \vee x = y)$
- $\forall x (x \neq 0 \rightarrow \exists y [y = \mathbf{S}(x)])$
- $\{\ \mathcal{T}(\overline{e},\overline{x},\overline{t}) \mid \mathbb{N} \models \mathcal{T}(e,x,t)\}$
- $\{\neg\mathcal{T}(\overline{e},\overline{x},\overline{t}) \mid \mathbb{N} \models \neg\mathcal{T}(e,x,t)\}$
- $\forall x \forall y \forall u \forall v \Big(\pi(x,y) = \pi(u,v) \iff x = u \wedge y = v\Big)$

Kurt Gödel's Incompleteness . . . COMPUTATIONALLY

Theory A_T is Consistent (and $\mathbb{N}-$Sound) but INCOMPLETABLE:

Let $\varphi_{\langle k,l\rangle} = \exists x[\mathcal{T}(\overline{k},\pi(\overline{k},\overline{l}),x) \wedge \forall y \leqslant x \neg\mathcal{T}(\overline{l},\pi(\overline{k},\overline{l}),y)]$.

If $\mathrm{T} \supseteq A_T$ is Complete (Not-Sound), then are Decidable: $\{\langle k,l\rangle \mid \mathrm{T} \vdash \varphi_{\langle k,l\rangle}\}$ and $\{\langle k,l\rangle \mid \mathrm{T} \vdash \neg\varphi_{\langle k,l\rangle}\}$.

Let Algorithm (with code) m on input $\langle k,l\rangle$ Halt Whenever $\mathrm{T} \vdash \varphi_{\langle k,l\rangle}$ and Never Halt Whenever $\mathrm{T} \nvdash \varphi_{\langle k,l\rangle}$.
Let Algorithm (with code) n on input $\langle k,l\rangle$ Halt Whenever $\mathrm{T} \vdash \neg\varphi_{\langle k,l\rangle}$ and Never Halt Whenever $\mathrm{T} \nvdash \neg\varphi_{\langle k,l\rangle}$.

Kurt Gödel's Incompleteness . . . COMPUTATIONALLY

Algorithm (with code) m on input $\langle k, l\rangle$ Halts Whenever $\mathrm{T}\vdash\varphi_{\langle k,l\rangle}$ and Never Halts Whenever $\mathrm{T}\nvdash\varphi_{\langle k,l\rangle}$.
Algorithm (with code) n on input $\langle k, l\rangle$ Halts Whenever $\mathrm{T}\vdash\neg\varphi_{\langle k,l\rangle}$ and Never Halts Whenever $\mathrm{T}\nvdash\neg\varphi_{\langle k,l\rangle}$.

Consider $\varphi_{\langle n,m\rangle}$: Was T Complete, then
either $\mathrm{T}\vdash\varphi_{\langle n,m\rangle}$ or $\mathrm{T}\vdash\neg\varphi_{\langle n,m\rangle}$.

We Will Get A Contradiction For Each Case ...

Kurt Gödel's Incompleteness . . . COMPUTATIONALLY

- If $\mathrm{T} \vdash \varphi_{\langle n,m\rangle}$ Then $\mathrm{T} \nvdash \neg\varphi_{\langle n,m\rangle}$. Thus $\mathcal{T}(m,\pi(n,m),t)$ holds for some t and $\neg\mathcal{T}(n,\pi(n,m),s)$ holds for every s. Also $\mathrm{T} \vdash \exists x[\mathcal{T}(\overline{n},\pi(\overline{n},\overline{m}),x) \wedge \forall y \leqslant x \neg\mathcal{T}(\overline{m},\pi(\overline{n},\overline{m}),y)]$. Since $\mathrm{T} \vdash \mathcal{T}(\overline{m},\pi(\overline{n},\overline{m}),\overline{t})$, then $x_0 < \overline{t}$. Whence, $\bigvee_{\{i<\overline{t}\}} x_0 = \overline{i}$, but then $A_T \vdash \bigwedge_{\{i<t\}} \neg\mathcal{T}(\overline{n},\pi(\overline{n},\overline{m}),\overline{i})$, so $\mathrm{T} \vdash \neg\mathcal{T}(\overline{n},\pi(\overline{n},\overline{m}),x_0)$. Contradiction!
- If $\mathrm{T} \vdash \neg\varphi_{\langle n,m\rangle}$ Then $\mathrm{T} \nvdash \varphi_{\langle n,m\rangle}$. Thus $\mathcal{T}(n,\pi(n,m),t)$ holds for some t and $\neg\mathcal{T}(m,\pi(n,m),s)$ holds for every s. Also $\mathrm{T} \vdash \forall x[\mathcal{T}(\overline{n},\pi(\overline{n},\overline{m}),x) \rightarrow \exists y \leqslant x\mathcal{T}(\overline{m},\pi(\overline{n},\overline{m}),y)]$. Since $A_T \vdash \mathcal{T}(\overline{n},\pi(\overline{n},\overline{m}),\overline{t})$, then $\mathrm{T} \vdash \mathcal{T}(\overline{m},\pi(\overline{n},\overline{m}),y_0)$ for some $y_0 \leqslant \overline{t}$. But then $\bigvee_{\{i\leqslant t\}} y_0 = \overline{i}$ and $A_T \subseteq \mathrm{T} \vdash \bigwedge_{\{i\leqslant t\}} \neg\mathcal{T}(\overline{m},\pi(\overline{n},\overline{m}),\overline{i})$. Contradiction!

Kurt Gödel's Incompleteness . . . COMPUTATIONALLY

▷ The Proof Resembles Rosser's Strengthening of
Gödel's Theorem for All Consistent Theories,
instead of Sound or $\omega-$Consistent Theories.

▷ The Proof is Effective:
For any (Hypothetical Code for) Enumeration of T, one can effectively find a (Gödel-Rosser) $T-$independent Sentence.

▷ Any Theory Capable of Interpreting A_T is INCOMPLETABLE
= Essentially Undecidable.
Like Robinson's Arithmetic *Q* or *PRA* or ...

▷ In the Proof Was Avoided:
Coding of Syntax (Coding of Algorithms Was Needed)
Constructing Gödel Sentence (`I Am Not Provable`)
Finding a Fixed Point Formula (`Diagonalization`)

Thank You!

Several Other Theorems Can Be Proved Similarly ...

PROBLEM: Find A Similar (Computational) Argument For Gödel's Second Incompleteness Theorem.

Thanks To
The Participants
and The Organizers of
IPM Weekly Seminars in Mathematical Logic