Logic and Computation, & their interactions

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Logic is . . .

- From the Greek word LOGOS, translated as "sentence", "discourse", "reason", "rule", and "ratio".
- The study of arguments (Wikipedia) "in the disciplines of philosophy, mathematics, and computer science".
- Logic is for constructing proofs which give us reliable confirmation of the truth of the proven proposition.

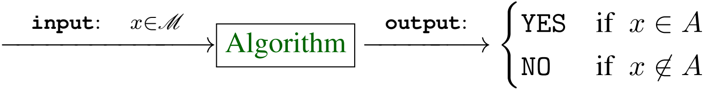
Mathematical Logic is . . .

- Application of mathematical techniques to logic.
- Mathematical logic (also known as symbolic logic) is a subfield of mathematics with close connections to computer science and philosophical logic. (Wikipedia)
- The field includes both the mathematical study of logic and the applications of formal logic to other areas of mathematics. (Wikipedia)

Hilbert's Entscheidungsproblem = Decision Problem

Finding an ALGORITHM (or AL-KHWARIZMI):
Input: A (Mathematical) Statement.
Output: *YES* (if universally valid) *NO* (if not always valid).

Computably Decidable set A: an algorithm $\mathcal{P}$ decides on any input x whether $x \in A$ (outputs **YES**) or $x \notin A$ (outputs **NO**).



Algorithm: single–input, Boolean–output (**1,0**)

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کتاب

جبر و مقابله

خوارزمی و میراث علمی وی

کتاب جبر و مقابله نخستین اثر علمی بر جای مانده است از محمد بن موسی خوارزمی، ریاضیدان بزرگ ایرانی، در علم حساب و جبر و مقابله. کتابی که در آغاز قرن سوم هجری - حدود یکهزار و دویست سال پیش از این - به همت و ابتکار این ریاضیدان ایرانی تبار به زبان عربی تصنیف شد و به فرهنگ رو به گسترش اسلامی رونق تازه بخشید. استقبال گرم و کم سابقه‌ای که در محافل علمی روزگار خوارزمی، و متعاقب قرنهای پس از وی، از این کتاب ریاضی به عمل آمد مهر تأییدی شد که بر نفاست کتاب و کفایت نویسنده‌اش علمی بحث و زمینهٔ یکتایی و جاودانگی نویسنده و نوشتهٔ او را در سراسر گیتی فراهم ساخت.

خوارزمی کار نگارش این اثر ماندنی خویش را در سال ۲۱۵ هجری به پایان رسانیده است. اثری که پس از انتشار در قلمرو جهان اسلام پیوسته استادان را مفید بوده و دانشجویان را کلید.

کتاب جبر خوارزمی در سال ۵۴۰ هجری (۱۱۴۵ میلادی) به همت رابرت چستری به لاتین ترجمه شد و این ترجمه را می‌توان آغاز رواج علم جبر در اروپا دانست و از آنجا به سراسر گیتی.

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کتاب

جبر و مقابله

نوشتهٔ

محمد بن موسی خوارزمی

ترجمهٔ

حسین خدیوجم

انتشارات اطلاعات

۱۳۶۳

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می کنی، می‌شود : شش درهم، و حاصل آن یك مال و یك جذر است که برابر است با شش درهم . آنگاه جذر را پس از نصف کردن، درمانند خودش ضرب کن ، می‌شود : یك چهارم، آن را برشش بیفزای، و جذر حاصل جمع را بگیر، و نصف جذری را که در مانند خودش ضرب کرده بودی ـ وعبارت است از نصف ـ از آن کم کن ، باقیمانده عبارت است از تعداد مردان نوبت اول که دراین مسئله دومرد است .

۲۹ـ اگر کسی بگوید: مالی است که چون آن را در دوسومش ضرب کنی پنج[1] می‌شود .

راه حل آن چنین است : اگر آنرا در مانند خودش ضرب کنی هفت و نیم می‌شود، پس می گویی: آنمال جذر هفت و نیم است که باید در دوسوم جذر هفت و نیم ضرب شود، آنگاه دوسوم را در دوسوم ضرب می کنی می‌شود چهار نهم، و چهارنهم ضرب درهفت ونیم می‌شود سه و یك سوم، پس جذر سه ویك سوم عبارت است از دوسوم جذر هفت ونیم، آنگاه سه و یك سوم را درهفت و نیم ضرب می کنی می‌شود بیست و پنج ، جذر آن را می گیری پنج می‌شود .

۳۰ـ اگر کسی بگوید : مالی است که چون درسه جذر خودش ضرب شود پنج برابر مال اول می‌شود .

راه حل آن چنین است: چنان است که گفته باشد مالی را درجذرش ضرب کردم به اندازهٔ یك مال و دوسوم مال اول شد ، پس مقدار جذر این مال یك درهم ودوسوم درهم است، و اصل مال دودرهم وهفت نهم درهم خواهد بود .

۳۱ـ اگر کسی بگوید: مالی است که چون یك سوم آن را کم

۱) خوارزمی این مسئله را با اندکی تفصیل تکرار کرده‌است . یعنی شکل دیگری از مسئلهٔ شمارهٔ ۱۲ است.

کنی و باقیمانده را درسه جذر آن مال ضرب کنی مقدار مال اول بدست می‌آید .

راه حل آن چنین است : اگر تمام مال اول را ، پیش از کسر یکسوم، درسه‌عدد جذرخودش ضرب کنی می‌شود یك مال ونیم؛ زیرا دو سوم آن ضرب در سه جذر خودش می‌شود یك مال ، پس تمام آن ضرب در سه جذرش می‌شود یك مال ونیم ، و چون تمام آنرا در یك جذر ضرب کنی می‌شود نصف مال، بنابراین جذر این مال نصف است واصل آن یك چهارم است ، پس دو سوم مال برابر است با یك ششم، وسه جذر مال یك درهم ونیم است، بنابراین هنگامی که یکششم را در یکونیم ضرب کنی یك چهارم بدست می آید و آن مقدار مال است

۳۲ ـ اگر کسی بگوید : مالی است که چون چهار جذر آن را کنار بگذاری وسپس یك سوم باقیمانده را برداری، این یکسوم برابر است باچهار جذر مال.

راه حل آن چنین است : میدانی که یك سوم باقیمانده برابر است با چهار جذر مال ، پس تمام باقیمانده برابر است با دوازده جذر آن ، و چون چهار جذری را که کنار گذاشتی بر آن بیفزایی می‌شود : شانزده جذر، و این تعداد جذر های مال است، و مقدار این مال دویست و پنجاه و شش است.

۳۳ ـ اگر کسی بگوید: مالی است که چون یك جذر آنرا کنار بگذاری وجذر باقیمانده را برجذر آن بیفزایی دو درهم می‌شود[1] .

راه حل آن چنین است : این معادله بدین صورت در می‌آید : جذر مال ، بهاضافهٔ جذر مال ، منهای یك جذر برابر است بادو درهم ، آنگاه یك جذر مال از آن و یك جذر مال از دو درهم کم می کنی ، معادله

۱) $x+\sqrt{x^۲-x}=۲$ بنابراین $x^۲-x=(۲-x)^۲$ تا آخر

ROBERT OF CHESTER'S

LATIN TRANSLATION

OF THE

ALGEBRA OF AL-KHOWARIZMI

WITH AN INTRODUCTION, CRITICAL NOTES AND AN ENGLISH VERSION

BY

LOUIS CHARLES KARPINSKI
UNIVERSITY OF MICHIGAN

Muhammad ibn Musa, al-Khowarazmi

New York
THE MACMILLAN COMPANY
LONDON: MACMILLAN AND COMPANY LIMITED
1915

equal to 6 units. I take one-half of the roots and I multiply the half by itself. I add the product to 6, and of this sum I take the root. The remainder obtained after subtracting one-half of the roots will designate the first number of girls, and this is two.

Fifteenth Problem

If from a square I subtract four of its roots and then take one-third of the remainder, finding this equal to four of the roots, the square will be 256.[1]

Explanation. Since one-third of the remainder is equal to four roots, you know that the remainder itself will equal 12 roots. Therefore add this to the four, giving 16 roots. This (16) is the root of the square.

Sixteenth Problem

From a square I subtract three of its roots and multiply the remainder by itself; the sum total of this multiplication equals the square.[2]

Explanation. It is evident that the remainder is equal to the root, which amounts to four. The square is 16.

These now are the sixteen problems which are seen to arise from the former ones, as we have explained. Hence by means of those things which have been set forth you will easily carry through any multiplication that you may wish to attempt in accordance with the art of restoration and opposition.

CHAPTER ON MERCANTILE TRANSACTIONS[3]

Mercantile transactions and all things pertaining thereto involve two ideas and four numbers.[4] Of these numbers the first is called by the Arabs Almuazahar and is the first one proposed. The second is called Alsiar, and recognized as second by means of the first. The third, Almuhen, is unknown. The fourth, Alchemon, is obtained by means of the first and second. Further, these four numbers are so related that the first of them, the measure, is inversely proportional to the last, which is cost. Moreover, three of these numbers are always given or known and the fourth is unknown, and this

[1] Rosen, p. 66; Libri, p. 296. $\frac{1}{3}(x^2 - 4x) = 4x$.
In the Arabic text these two problems precede: [illegible]
[2] Rosen, p. 67; Libri, p. 296. $(x^2 - 3x)^2 = x^2$, whence $x^2 - 3x = x$.
The problem, [illegible], precedes. This is one of two problems given in the German excerpt of 1461 from the algebra of Al-Khowarizmi (Gerhardt, *Monatsberichte d. Königl. Akad. der Wissenschaften zu Berlin*, 1870, pp. 141–143).
[3] The famous "rule of three" is the subject of discussion in this chapter.
[4] The two ideas appear to be the notions of quantity and cost; the four numbers represent unit of measure and price per unit, quantity desired and cost of the same. These four technical terms are [illegible], and further [illegible]; see p. 44.

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Coding Mathematics

How to write (code) mathematical statements (as input strings)?

Example from Al-Khwarizmi: If from a square, I subtract four of its roots and then take one-third of the remainder, finding this equal to four of the roots, the square will be 256.

Modern Notation: If I have $\frac{1}{3}(x^2 - 4x) = 4x$, then $x^2 = 256$.

More Modern: $\forall x[\frac{1}{3}(x^2 - 4x) = 4x \longrightarrow x^2 = 256]$.

This holds in the domain $\mathbb{N} - \{0\} = \{1, 2, 3, \cdots\}$ (but not in $\mathbb{N}$).

Indeed, $\mathbb{N} \models \forall x[\frac{1}{3}(x^2 - 4x) = 4x \longrightarrow x = 16 \vee x = 0]$.

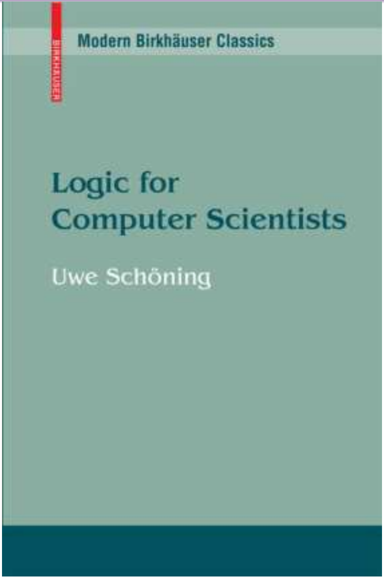
Computing the Solution

Khwarizmi's Explanation:

Since one-third of the remainder is equal to four roots, you know that the remainder itself will equal 12 roots. Therefore, add this to the four, giving 16 roots. This (16) is the root of the square.

Modern Notation: Since $\frac{1}{3}(x^2 - 4x) = 4x$, then $x^2 - 4x = 12x$.
Therefore, $x^2 = 16x$. Thus, $x = 16$.

In fact, $\mathfrak{Arithmetic} \vdash \forall x \big[\frac{1}{3}(x^2 - 4x) = 4x \ (\& x \neq 0) \longrightarrow x = 16\big]$.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Logic for Computer Scientists

Uwe Schöning

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Logic for Computer Scientists

With 34 Illustrations

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Preface

By the development of new fields and applications, such as Automated Theorem Proving and Logic Programming, Logic has obtained a new and important role in Computer Science. The traditional mathematical way of dealing with Logic is in some respect not tailored for Computer Science applications. This book emphasizes such Computer Science aspects in Logic. It arose from a series of lectures in 1986 and 1987 on Computer Science Logic at the EWH University in Koblenz, Germany. The goal of this lecture series was to give the undergraduate student an early and theoretically well-founded access to modern applications of Logic in Computer Science.

A minimal mathematical basis is required, such as an understanding of the set theoretic notation and knowledge about the basic mathematical proof techniques (like induction). More sophisticated mathematical knowledge is not a precondition to read this book. Acquaintance with some conventional programming language, like PASCAL, is assumed.

Several people helped in various ways in the preparation process of the original German version of this book: Johannes Köbler, Eveline and Rainer Schuler, and Hermann Engesser from B.I. Wissenschaftsverlag.

Regarding the English version, I want to express my deep gratitude to Prof. Ronald Book. Without him, this translated version of the book would not have been possible.

Koblenz, June 1989 U. Schöning

Contents

Introduction

Formal Logic investigates how assertions are combined and connected, how theorems formally can be deduced from certain axioms, and what kind of object a proof is. In Logic there is a consequent separation of syntactical notions (formulas, proofs) – these are essentially strings of symbols built up according to certain rules – and semantical notions (truth values, models) – these are "interpretations", assignments of "meanings" to the syntactical objects.

Because of its development from philosophy, the questions investigated in Logic were originally of a more fundamental character, like: What is truth? What theories are axiomatizable? What is a model of a certain axiom system?, and so on. In general, it can be said that traditional Logic is oriented to fundamental questions, whereas Computer Science is interested in what is programmable. This book provides some unification of both aspects.

Computer Science has utilized many subfields of Logic in areas such as program verification, semantics of programming languages, automated theorem proving, and logic programming. This book concentrates on those aspects of Logic which have applications in Computer Science, especially theorem proving and logic programming. From the very beginning, education in Computer Science supports the idea of strict separation between syntax and semantics (of programming languages). Also, recursive definitions, equations and programs are a familiar thing to a first year Computer Science student. This book is oriented in its style of presentation to this style.

In the first Chapter, propositional logic is introduced with emphasis on the resolution calculus and Horn formulas (which have their particular Computer Science applications in later sections). The second Chapter introduces the predicate logic. Again, Computer Science aspects are emphasized, like undecidability and semi-decidability of predicate logic, Herbrand's the-

ory, and building upon this, the resolution calculus (and its refinements) for predicate logic is discussed. Most modern theorem proving programs are based on resolution refinements as discussed in Section 2.6.

The third Chapter leads to the special version of resolution (SLD-resolution) used in logic programming systems, as realized in the logic programming language PROLOG (= *Programming in Logic*). The idea of this book, though, is not to be a programmer's manual for PROLOG. Rather, the aim is to give the theoretical foundations for an understanding of logic programming in general.

Exercise 1: "What is the secret of your long life?" a centenarian was asked. "I strictly follow my diet: If I don't drink beer for dinner, then I always have fish. Any time I have both beer and fish for dinner, then I do without ice cream. If I have ice cream or don't have beer, then I never eat fish." The questioner found this answer rather confusing. Can you simplify it?

Find out which formal methods (diagrams, graphs, tables, etc.) you used to solve this Exercise. You have done your own first steps to develop a Formal Logic!

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Proving or Computing?

Exercise 1: "What is the secret of your long life?"
a centenarian was asked.

"I strictly follow my diet:
If I don't drink beer for dinner, then I always have fish.
If I have both beer and fish for dinner, then I do without ice cream.
If I have ice cream or don't have beer, then I never eat fish."

The questioner found this answer rather confusing.
Can you simplify it?

Proving or Computing?

$B =$ beer $\quad F =$ fish $\quad I =$ ice cream

If I don't drink beer for dinner, then I always have fish.

$$\neg B \to F$$

Any time I have both beer and fish for dinner, then I do without ice cream.

$$B \wedge F \to \neg I$$

If I have ice cream or don't have beer, then I never eat fish.

$$I \vee \neg B \to \neg F$$

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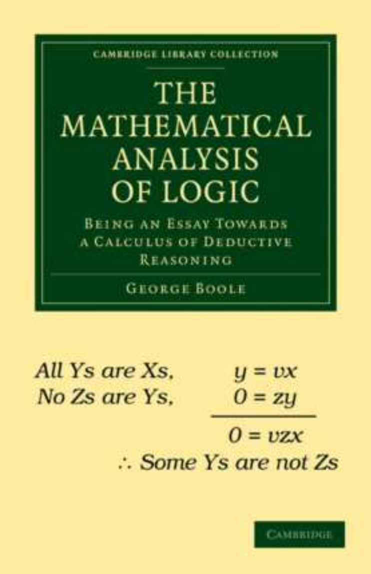

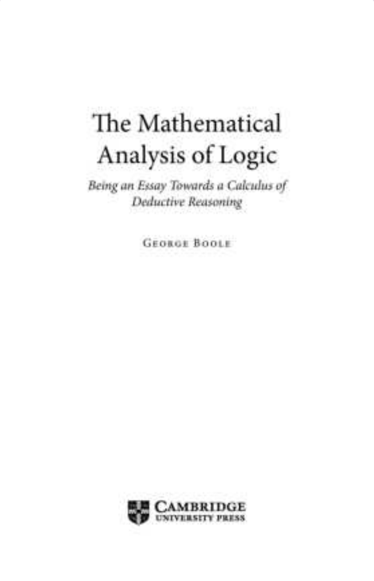

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THE MATHEMATICAL ANALYSIS OF LOGIC,

BEING AN ESSAY TOWARDS A CALCULUS OF DEDUCTIVE REASONING.

BY GEORGE BOOLE.

Ἐπικοινωνοῦσι δὲ πᾶσαι αἱ ἐπιστῆμαι ἀλλήλαις κατὰ τὰ κοινά. Κοινὰ δὲ λέγω, οἷς χρῶνται ὡς ἐκ τούτων ἀποδεικνύντες· ἀλλ᾽ οὐ περὶ ὧν δεικνύουσιν, οὐδὲ ὃ δεικνύουσι.

ARISTOTLE, *Anal. Post.*, lib. i. cap. xi.

CAMBRIDGE:
MACMILLAN, BARCLAY, & MACMILLAN;
LONDON: GEORGE BELL.

1847

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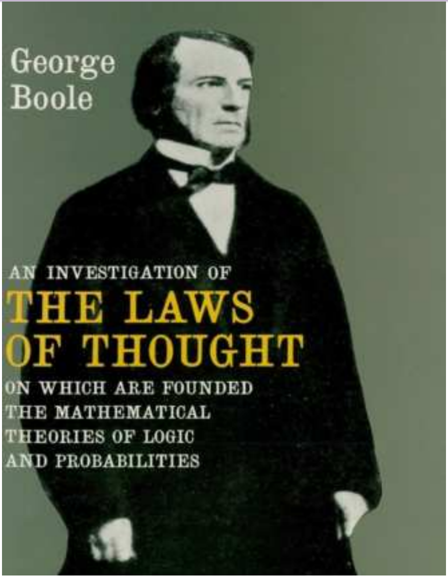
George
Boole
AN INVESTIGATION OF
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GEORGE BOOLE, L.L.D.

DOVER PUBLICATIONS, INC., NEW YORK

Propositional Logic

- Connectives $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$
- Atomic Propositions (without a truth value) $P, Q, R, \cdots$
- More Complex Propositions and Truth Tables

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Proving or Computing?

$$(\neg B \rightarrow F), (B \wedge F \rightarrow \neg I), (I \vee \neg B \rightarrow \neg F)$$
$$\varphi = (\neg B \rightarrow F) \wedge (B \wedge F \rightarrow \neg I) \wedge (I \vee \neg B \rightarrow \neg F)$$

B	F	I	$\neg B \rightarrow F$	$B \wedge F \rightarrow \neg I$	$I \vee \neg B \rightarrow \neg F$	φ
0	0	0	0	1	1	0
0	0	1	0	1	1	0
0	1	0	1	1	0	0
0	1	1	1	1	0	0
1	0	0	1	1	1	1
1	0	1	1	1	1	1
1	1	0	1	1	1	1
1	1	1	1	0	0	0

https://web.stanford.edu/class/cs103/tools/truth-table-tool/

Proving or Computing?

B	F	I	$\neg B \to F$	$B \wedge F \to \neg I$	$I \vee \neg B \to \neg F$	φ
1	0	0	1	1	1	1
1	0	1	1	1	1	1
1	1	0	1	1	1	1

$\varphi \equiv$
$(B \wedge \neg F \wedge \neg I) \vee$
$(B \wedge \neg F \wedge I) \vee$
$(B \wedge F \wedge \neg I)$

$$\equiv (B \wedge \neg F) \vee (B \wedge F \wedge \neg I) \equiv B \wedge (\neg F \vee [F \wedge \neg I]) \equiv B \wedge (\neg F \vee \neg I)$$

$$\varphi \equiv B \wedge \neg (F \wedge I)$$

Axiom / Axiomatic / Axiomaitze

Merriam-Webster: www.merriam-webster.com

AXIOM:

a statement accepted as true as the basis for argument or inference

Postulate

AXIOMATIC:

based on or involving an axiom or system of axioms

AXIOMATIZATION:

the act or process of reducing to a system of axioms

Axiom / Axiomatic / Axiomaitze

Oxford: www.oxforddictionaries.com

AXIOM:

a statement or proposition which is regarded as being established, accepted, or self-evidently true *the axiom that sport builds character*

Math: a statement or proposition on which an abstractly defined structure is based

Origin: late 15th century: from French axiome or Latin axioma, from Greek axio-ma 'what is thought fitting', from axios 'worthy'

AXIOMATIC: self-evident or unquestionable

it is axiomatic that good athletes have a strong mental attitude

Math: relating to or containing axioms

AXIOMATIZE: express (a theory) as a set of axioms

the attempts that are made to axiomatize linguistics

Algebraic Axiomatizing "The Laws of Thought"

Language: $\bot, \top$ $\neg$ $\wedge, \vee$ $\equiv$

Idempotence:	$p \wedge p \equiv p$	$p \vee p \equiv p$
Commutativity:	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associativity:	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	$p \vee (q \vee r) \equiv (p \vee q) \vee r$
Distributivity:	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	
Distributivity:	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	
Tautology:	$p \wedge \top \equiv p$	$p \vee \top \equiv \top$
Contradiction:	$p \wedge \bot \equiv \bot$	$p \vee \bot \equiv p$
Negation:	$p \wedge (\neg p) \equiv \bot$	$p \vee (\neg p) \equiv \top$
Negation:		$\neg(\neg p) \equiv p$
DeMorgan:	$\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$	$\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$

Computing the PROOF!

$$\varphi = (\neg B \rightarrow F) \wedge (B \wedge F \rightarrow \neg I) \wedge (I \vee \neg B \rightarrow \neg F)$$

$$\equiv (B \vee F) \wedge (\neg B \neg \vee F \vee \neg I) \wedge ([B \wedge \neg I] \vee \neg F)$$

$$\equiv (B \vee F) \wedge (\neg B \vee \neg F \vee \neg I) \wedge (B \vee \neg F) \wedge (\neg I \vee \neg F)$$

$$\equiv (B \vee F) \wedge (B \vee \neg F) \wedge (\neg B \vee \neg I \vee \neg F) \wedge (\neg I \vee \neg F)$$

$$\equiv (B \vee [F \wedge \neg F]) \qquad \wedge \qquad (\neg I \vee \neg F)$$

$$\equiv B \wedge \neg (I \wedge F)$$

https://www.wolframalpha.com/

Axiomatizing Propositional Logic

AX_1 $\alpha \to (\beta \to \alpha)$

AX_2 $[\alpha \to (\beta \to \gamma)] \to [(\alpha \to \beta) \to (\alpha \to \gamma)]$

AX_3 $(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$

RUL $\dfrac{\alpha, \quad \alpha \to \beta}{\beta}$

Some Theorems (EXERCISES):

$\alpha \to \alpha$

$(\neg\beta) \to (\beta \to \alpha)$

$(\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$

$[(\alpha \to \beta) \to \alpha] \to \alpha$

Predicate Logic

- Quantifiers $\forall$, $\exists$
- A Language of Undefined Relations or Functions
  (or Constants)
- More Complex Propositions and Models
  (Complicated Algebraic Structures)

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Axiomatizing Predicate Logic

Gödel's Completeness Theorem (1929)

From An Axiomatization of (Logically) Valid Formulas:

- $\alpha \to (\beta \to \alpha)$
- $(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$
- $[\alpha \to (\beta \to \gamma)] \to [(\alpha \to \beta) \to (\alpha \to \gamma)]$
- $\forall x \varphi(x) \to \varphi(t)$
- $\varphi \to \forall x \varphi$ [x is not free in φ]
- $\forall x(\varphi \to \psi) \to (\forall x \varphi \to \forall x \psi)$

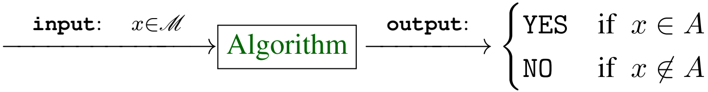
With the Modus Ponens Rule:

- $\dfrac{\varphi, \quad \varphi \to \psi}{\psi}$

All the Universally Valid Formulas CAN BE GENERATED.

Computably Decidable Set

Computably Decidable set A: an algorithm $\mathcal{P}$ decides on any input x whether $x \in A$ (outputs **YES**) or $x \notin A$ (outputs **NO**).



Algorithm: single–input, Boolean–output (**1,0**)

Propositional Logic is DECIDABLE.

Algorithms: Truth-Tables, Various Deductive Calculi, etc.
Now the question is the speed of algorithms ...

Computably Enumerable Set

Computably Enumerable set A: an (input-free) algorithm $\mathcal{P}$ lists all members of A; i.e., $A = \texttt{output}(\mathcal{P})$.

$$\boxed{\text{Algorithm}} \xrightarrow{\texttt{output}:} \{a_0, a_1, a_2, \cdots\} = A$$

Algorithm: input–free, outputs a set.

Predicate Logic is COMPUTABLY ENUMERABLE (GÖDEL 1929).

Predicate Logic is NOT DECIDABLE (CHURCH & TURING 1936).

► A Good Outcome: Introducing Turing Machines
– the grand grandfather of today's modern computers.

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Decision Problem, again

Decision Problem for the Structure $(\mathfrak{M}, \mathcal{L})$:

Input: A First–Order Sentence φ in the Language $\mathcal{L}$.
Output: *YES* (if $\mathfrak{M} \models \varphi$) *NO* (if $\mathfrak{M} \not\models \varphi$).

Examples:

- $\mathbb{N} \not\models \forall x \exists y(x + y = 0)$ but $\mathbb{Z} \models \forall x \exists y(x + y = 0)$.
- $\mathbb{Z} \not\models \forall x \exists y(x \neq 0 \rightarrow [x \cdot y = 1])$ but $\mathbb{Q} \models \forall x \exists y(x \neq 0 \rightarrow [x \cdot y = 1])$.
- $\mathbb{Q} \not\models \forall x \exists y(0 \leqslant x \rightarrow [y \cdot y = x])$ but $\mathbb{R} \models \forall x \exists y(0 \leqslant x \rightarrow [y \cdot y = x])$.
- $\mathbb{R} \not\models \forall x \exists y(y \cdot y + x = 0)$ but $\mathbb{C} \models \forall x \exists y(y \cdot y + x = 0)$.

Decidability of Mathematical Structures

The Decidability Problem for the Structures:

	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$	$\mathbb{C}$
$\{<\}$	$\langle\mathbb{N};<\rangle$	$\langle\mathbb{Z};<\rangle$	$\langle\mathbb{Q};<\rangle$	$\langle\mathbb{R};<\rangle$	—
$\{+\}$	$\langle\mathbb{N};+\rangle$	$\langle\mathbb{Z};+\rangle$	$\langle\mathbb{Q};+\rangle$	$\langle\mathbb{R};+\rangle$	$\langle\mathbb{C};+\rangle$
$\{\cdot\}$	$\langle\mathbb{N};\cdot\rangle$	$\langle\mathbb{Z};\cdot\rangle$	$\langle\mathbb{Q};\cdot\rangle$	$\langle\mathbb{R};\cdot\rangle$	$\langle\mathbb{C};\cdot\rangle$
$\{+,<\}$	$\langle\mathbb{N};+,<\rangle$	$\langle\mathbb{Z};+,<\rangle$	$\langle\mathbb{Q};+,<\rangle$	$\langle\mathbb{R};+,<\rangle$	—
$\{+,\cdot\}$	$\langle\mathbb{N};+,\cdot\rangle$	$\langle\mathbb{Z};+,\cdot\rangle$	$\langle\mathbb{Q};+,\cdot\rangle$	$\langle\mathbb{R};+,\cdot\rangle$	$\langle\mathbb{C};+,\cdot\rangle$
$\{\cdot,<\}$	$\langle\mathbb{N};\cdot,<\rangle$	$\langle\mathbb{Z};\cdot,<\rangle$	$\langle\mathbb{Q};\cdot,<\rangle$	$\langle\mathbb{R};\cdot,<\rangle$	—
$\{+,\cdot,<\}$	$\langle\mathbb{N};+,\cdot,<\rangle$	$\langle\mathbb{Z};+,\cdot,<\rangle$	$\langle\mathbb{Q};+,\cdot,<\rangle$	$\langle\mathbb{R};+,\cdot,<\rangle$	—
E	$\langle\mathbb{N};\exp\rangle$	—	—	$\langle\mathbb{R};+,\cdot,e^x\rangle$	$\langle\mathbb{C};+,\cdot,e^x\rangle$

New Results

- SALEHI, SAEED; *On Axiomatizability of the Multiplicative Theory of Numbers*, **Fundamenta Informaticæ** 159:3 (**2018**) 279–296.

  $\langle\mathbb{Q};\times\rangle$, $\langle\mathbb{R};\times\rangle$, $\langle\mathbb{C};\times\rangle$.

- ASSADI, ZIBA & SALEHI, SAEED; *On Decidability and Axiomatizability of Some Ordered Structures*, **Soft Computing** 23:11 (**2019**) 3615–3626.

  $\langle\mathbb{Q};\times,<\rangle$, $\langle\mathbb{R};\times,<\rangle$.

- SALEHI, SAEED & ZARZA, MOHAMMADSALEH; *First-Order Continuous Induction and a Logical Study of Real Closed Fields*, **Bulletin of the Iranian Mathematical Society** online (**2019**) DOI: 10.1007/s41980-019-00252-0.

  $\langle\mathbb{R};+,\times,<\rangle$.

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On decidability and axiomatizability of some ordered structures
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Axiomatizability of Mathematical Structures

A Rather Complete Picture

	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$	$\mathbb{C}$
$\{<\}$	Δ_1	Δ_1	Δ_1	Δ_1	–
$\{+\}$	Δ_1	Δ_1	Δ_1	Δ_1	Δ_1
$\{\cdot\}$	Δ_1	Δ_1	Δ_1	Δ_1	Δ_1
$\{+,<\}$	Δ_1	Δ_1	Δ_1	Δ_1	–
$\{+,\cdot\}$	$\not\Delta_1$	$\not\Delta_1$	$\not\Delta_1$	Δ_1	Δ_1
$\{\cdot,<\}$	$\not\Delta_1$	$\not\Delta_1$	Δ_1	Δ_1	–
$\{+,\cdot,<\}$	$\not\Delta_1$	$\not\Delta_1$	$\not\Delta_1$	Δ_1	–
E	$\not\Delta_1$	–	–	¿?	$\not\Delta_1$

Tarski's Exponential Function Problem **is open ...**

Is the structure $\langle\mathbb{R};+,\cdot,e^x\rangle$ decidable?

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Thank You!

Thanks to

The Participants ....For Listening...

and

The Organizers ....For Taking Care of Everything...

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