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Gödel's Incompleteness Phenomenon—Computationally

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Résumé : Nous soutenons que le théorème de complétude de Gödel est équivalent à la complétabilité de théories consistentes, et que son théorème d'incomplétude est équivalent au fait que cette complétion n'est pas constructive, en ce sens qu'il existe des théories consistantes et récursivement énumérables qui ne peuvent être étendues à aucune théorie complète et consistente et récursivement énumérable. Toutefois, n'importe quelle théorie consistente et décidable peut être étendue à une théorie complète, consistente et décidable. Donc la déduction et la consistence ne sont pas décidables en logique, et un analogue du théorème de Rice est valable pour les théories récursivement énumérables : toutes leurs propriétés non-triviales sont indécidables.

Abstract: We argue that Gödel's completeness theorem is equivalent to completability of consistent theories, and Gödel's incompleteness theorem is equivalent to the fact that this completion is not constructive, in the sense that there are some consistent and recursively enumerable theories which cannot be extended to any complete and consistent and recursively enumerable theory. Though any consistent and decidable theory can be extended to a complete and consistent and decidable theory. Thus deduction and consistency are not decidable in logic, and an analogue of Rice's Theorem holds for recursively enumerable theories: all the non-trivial properties of them are undecidable.

1 Introduction

The incompleteness theorem of Kurt Gödel has been regarded as the most significant mathematical result in the twentieth century, and Gödel's completeness theorem is a kind of the fundamental theorem of mathematical logic.

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To avoid confusion between these two results, it is argued in the literature that the completeness theorem is about the *semantic* completeness of first order logic, and the incompleteness theorem is about the *syntactic* incompleteness of sufficiently strong first order logical theories. In this paper we look at these two theorems from another perspective. We will argue that Gödel's completeness theorem is a kind of completability theorem, and Gödel-Rosser's incompleteness theorem is a kind of incompletability theorem in a constructive manner. By Gödel's semantic incompleteness theorem we mean the statement that *any sound and sufficiently strong and recursively enumerable theory is incomplete.* By Gödel-Rosser's incompleteness theorem we mean the statement that *any consistent and sufficiently strong and recursively enumerable theory is incomplete.* Gödel's original incompleteness theorem's assumption is between soundness and consistency; it assumes $\omega-$consistency of sufficiently strong and recursively enumerable theories which are to be proved incomplete.

It is noted in the literature that the existence of a non-recursive but recursively enumerable set can prove Gödel's semantic incompleteness theorem (see e.g., [Lafitte 2009] or [Li & Vitányi 2008]). This beautiful proof is most likely first proposed by [Kleene 1936] and Church; below we will give an account of this proof after Theorem 12. A clever modification of this proof shows Gödel-Rosser's (stronger) incompleteness theorem, and in fact provides an elementary and nice proof of Gödel-Rosser's theorem other than the classical Rosser's trick [Rosser 1936]. This is called *Kleene's Symmetric Form of Gödel's Incompleteness Theorem* (see [Beklemishev 2010]) originally published in [Kleene 1950] and later in the book [Kleene 1952]. Indeed, Gödel's semantic incompleteness theorem is equivalent to the existence of a non-recursive but recursively enumerable set, and also Gödel-Rosser's (constructive) incompleteness theorem is equivalent to the existence of a pair of recursively (effectively) inseparable recursively enumerable sets.

We will present a theory which is computability theoretic in nature, in a first order language which does not contain any arithmetical operations like addition or multiplication, nor set theoretic relation like membership nor sting theoretic operation like concatenation. We will use a ternary relation symbol τ which resembles Kleene's T predicate and our theory resembles Robinson's $\mathtt{R}$ arithmetic (see [Tarski, Mostowski *et al.* 1953]). The proofs avoid using the diagonal (or fixed-point) lemma which is highly counter-intuitive and a kind of "pulling a rabbit out of the hat" (see [Wasserman 2008]); the proofs are also constructive, in the sense that given a recursively enumerable theory that can interpret our theory one can algorithmically produce an independent sentence. For us the simplicity of the proofs and elementariness of the arguments are of essential importance. Though we avoid coding sentences and proofs and other syntactic notions, coding programs is needed for interpreting the τ relation. We also do not need any mathematical definition for algorithms or programs (like recursive functions or Turing machines etc); all we need is the finiteness of programs (every program is a finite string of ASCII[1] codes) and the finiteness of input and time of computation (which can be coded or measured by natural numbers). So, Church's Thesis (that every intuitively computable function is a recursive function, or a function defined rigorously in a mathematical framework) is not used in the arguments.

2 Completeness and completability

In mathematical logic, a *theory* is said to be a set of sentences, in a fixed language (see e.g., [Chiswell & Hodges 2007]; in [Kaye 2007] for example the word "theory" does not appear in this sense at all, and instead "a set of sentences" is used). Sometimes a theory is required to be closed under (logical) deduction, i.e., a set of sentences T is called a theory if for any sentence φ which satisfies $T \vdash \varphi$ we have $\varphi \in T$ (see e.g., [Enderton 2001]). Here, by a *theory* we mean any set of sentences (not necessarily closed under deduction). Syntactic completeness of a theory is usually taken to be *negation*-completeness: a theory T is *complete* when for any sentence φ, either $T \vdash \varphi$ or $T \vdash \neg\varphi$. Let us look at the completeness with respect to other connectives:

Definition 1 (Completeness)**.** A theory T is called

- $\neg-$complete when for any sentence φ:
$$T \vdash \neg\varphi \iff T \nvdash \varphi.$$
- $\wedge-$complete when for any sentences φ and ψ:
$$T \vdash \varphi \wedge \psi \iff T \vdash \varphi \text{ and } T \vdash \psi.$$
- $\vee-$complete when for any sentences φ and ψ:
$$T \vdash \varphi \vee \psi \iff T \vdash \varphi \text{ or } T \vdash \psi.$$
- $\rightarrow$ $-$complete when for any sentences φ and ψ:
$$T \vdash \varphi \rightarrow \psi \iff \text{if } T \vdash \varphi \text{ then } T \vdash \psi.$$
- $\forall-$complete when for every formula $\varphi(x)$:
$$T \vdash \forall x \varphi(x) \iff \text{for every } t, T \vdash \varphi(t).$$
- $\exists-$complete when for every formula $\varphi(x)$:
$$T \vdash \exists x \varphi(x) \iff \text{for some } t, T \vdash \varphi(t). \qquad \diamond$$

Let us note that the half of $\neg-$completeness is *consistency*: a theory is called *consistent* when for every sentence φ, if $T \vdash \neg\varphi$ then $T \nvdash \varphi$. Usually, the other half is called *completeness*, i.e., when if $T \nvdash \varphi$ then $T \vdash \neg\varphi$ for every sentence φ.

Remark 2. *Any theory is $\wedge-$complete and $\forall-$complete (in first-order logic). Also, half of $\vee, \rightarrow, \exists-$completeness holds for all theories T; i.e.,*

— if $T \vdash \varphi$ or $T \vdash \psi$, then $T \vdash \varphi \vee \psi$;

[1] `www.ascii-code.com/`.

— *if* $T \vdash \varphi \rightarrow \psi$*, then if* $T \vdash \varphi$ *then* $T \vdash \psi$*;*

— *if* $T \vdash \varphi(t)$ *for some* t*, then* $T \vdash \exists x\varphi(x)$*.* ◇

A *maximally consistent* theory is a theory T which cannot properly be extended to a consistent theory; i.e., for any consistent theory T' which satisfies $T \subseteq T'$ we have $T = T'$. The following is a classical result in mathematical logic (see e.g., [van Dalen 2013]).

Remark 3. *A consistent theory is* $\neg-$*complete if and only if is* $\vee-$*complete if and only if is* $\rightarrow -$*complete if and only if is maximally consistent.* ◇

Consistently maximizing a theory suggests using Zorn's Lemma or (equivalently) the Axiom of Choice, which is non-constructive in general. To see if one can do it constructively or not, we need to introduce some other notions. Before that let us note that $\exists-$completing a theory can be done constructively.

Remark 4. *Any arbitrary first-order consistent theory can be extended (constructively) to another consistent* $\exists-$*complete theory.* ◇

The main idea of the proof is that we add a countable set of constants $\{c_1, c_2, \cdots\}$ to the language, and then enumerate all the couples of formulas and variables in the extended language as $\langle\varphi_1, x_1\rangle, \langle\varphi_2, x_2\rangle, \cdots$ and finally add the sentences $\exists x_1\varphi_1 \rightarrow \varphi(c_{l_1}/x_1)$, $\exists x_2\varphi_2 \rightarrow \varphi(c_{l_2}/x_2)$, $\cdots$ successively to the theory, where in each step c_{l_i} is the first constant which does not appear in $\varphi_1, \ldots, \varphi_i$ and has not been used in earlier steps (see e.g., [Enderton 2001]).

Let us note that $\exists-$complete theories are sometimes called Henkin theories or Henkin-complete or Henkin sets (see e.g., [van Dalen 2013]). These are used for proving Gödel's Completeness Theorem by Henkin's proof. The theory of a structure is the set of sentences (in the language of that structure) which are true in that structure. It can be seen that theories of structures are $(\neg, \exists)-$complete theories. Conversely, for any $(\neg, \exists)-$complete theory T one can construct a structure $\mathcal{M}$ such that T is the theory of $\mathcal{M}$.

Remark 5. *Any consistent theory can be extended to a* $(\neg, \exists)-$*complete theory. Note that any* $(\neg, \exists)-$*complete theory is complete with respect to all the other connectives.* ◇

Gödel's completeness theorem is usually proved by showing that any consistent theory has a model (the model existence theorem–which is equivalent to the original completeness theorem). Note that for proving the model existence theorem it is shown that any consistent theory is extendible to a consistent $(\neg, \exists)-$complete theory, which then defines a structure which is a model of that theory. Thus, we can rephrase this theorem equivalently as follows.

GÖDEL'S COMPLETENESS THEOREM: *Any first-order consistent theory can be extended to a consistent* $(\neg, \exists)-$*complete theory.*

This theorem can be considered as *the fundamental theorem of logic*, the same way that we have the fundamental theorem of arithmetic, or the fundamental theorem of algebra, or the fundamental theorem of calculus. We could also call this theorem, Gödel's *Completability* Theorem, for the above reasons.

3 Incompleteness and incompletability

We now turn our attention to constructive aspects of the above theorem. A possibly infinite set can be constructive when it is *decidable* or (at least) *recursively enumerable.* A set D is decidable when there exists a single-input algorithm which on any input x outputs `Yes` if $x \in D$ and outputs `No` if $x \notin D$. A set R is called recursively enumerable (RE for short) when there exists an input-free algorithm which outputs (generates) the elements of R (after running). It is a classical result in Computability Theory that there exists an RE set which is not decidable (though, any decidable set is RE); see e.g., [Epstein & Carnielli 2008]. For a theory T we can consider decidability or recursive enumerability of either T as a set of sentences, or the set of derivable sentences of T, i.e., $\mathrm{Der}(T) = \{\varphi \mid T \vdash \varphi\}$. It can be shown that if T is decidable or RE (as a set) then $\mathrm{Der}(T)$ is RE; of course when $\mathrm{Der}(T)$ is RE then T is RE as well, and by Craig's trick [Craig 1953] for such a theory there exists a decidable set of sentences $\widehat{T}$ such that $\mathrm{Der}(T) = \mathrm{Der}(\widehat{T})$. So, we consider RE theories only, and call theory T a *decidable theory* when $\mathrm{Der}(T)$ is a decidable set (of sentences). RE theories are sometimes called axiomatizable theories (in e.g., [Enderton 2001]). Below we will show that there exists some decidable set of sentences ($\mathcal{T}$) whose set of derivable sentences is not decidable (though it must be RE of course).

Definition 6 (RE, Decidable and RE-Completable)**.** A consistent theory

- T is called an RE *theory* when $\mathrm{Der}(T)$ is an RE set.
- T is called a *decidable* theory when $\mathrm{Der}(T)$ is a decidable set.
- T is called RE-*completable* when there exists a theory T' extending T (i.e., $T \subseteq T'$) such that T' is consistent, complete and RE. ◇

It is a classical fact that complete RE theories are decidable (see e.g., [Enderton 2001]): since by recursive enumerability of T both $\{\varphi \mid T \vdash \varphi\}$ and $\{\varphi \mid T \vdash \neg\varphi\}$ are RE and by the completeness of T we have $\{\varphi \mid T \nvdash \varphi\} = \{\varphi \mid T \vdash \neg\varphi\}$, so the set $\mathrm{Der}(T)$ and its complement are both RE and hence decidable (by Kleene's Complementation Theorem—see e.g., [Berto 2009]). Completeness is a logician's tool for decidability. Henkin's completion shows that any RE decidable theory is RE—completable, see [Tarski, Mostowski *et al.* 1953]. The main idea is that having a decidable theory T we list all the sentences in the language of T as $\varphi_1, \varphi_2, \cdots$ and then add φ_i or $\neg\varphi_i$ in the ith step to T as follows: let $T_0 = T$ and if T_j is defined let $T_{j+1} = T_j \cup \{\varphi_j\}$ if

$T_j \cup \{\varphi_j\}$ is consistent, otherwise let $T_{j+1} = T_j \cup \{\neg\varphi_j\}$. Note that if T_j is consistent, then T_{j+1} will be consistent as well (if $T_j \cup \{\varphi_j\}$ is inconsistent then $T_j \cup \{\neg\varphi_j\}$ must be consistent). The theory $T' = \bigcup_{i \geqslant 0} T_i$ will be consistent and complete. This was essentially Henkin's Construction for proving Gödel's completeness theorem. The point is that if T is decidable then so is any T_i since they are finite extensions of T. Finally, T' is a decidable theory, because for any given sentence φ it should appear in the list $\varphi_1, \varphi_2, \cdots$, so say $\varphi = \varphi_n$. Now, for $i = 1, 2, \ldots, n$ we can decide whether $\varphi_i \in T'$ or $\neg\varphi_i \in T'$ inductively; and finally we can decide whether $T' \vdash \varphi$ or not ($T \nvdash \varphi$ happens only when $T' \vdash \neg\varphi$). So, for consistent RE theories we have the following inclusions:

Complete $\Longrightarrow$ Decidable $\Longrightarrow$ RE-Completable

Below we will see that the converse conclusions do not hold (Remark 10). Whence, by contrapositing the above conclusions we will have the following inclusions and non-inclusions for some consistent RE theories:

RE-Incompletable $\overset{\Longrightarrow}{\not\Longleftarrow}$ Undecidable $\overset{\Longrightarrow}{\not\Longleftarrow}$ Incomplete

Incomplete theories abound in mathematics: every theory which has finite models but does not fix the number of elements (e.g., theory of groups, rings, fields, lattices, etc.) is an incomplete theory. By encoding Turing machines into a first order language one can obtain an undecidable theory (see e.g., [Boolos, Burgess *et al.* 2007]). But demonstrating an RE-incompletable theory is a difficult task and it is in fact Gödel's Incompleteness Theorem. RE-incompletable theories are known as "essentially undecidable" theories in the literature (starting from [Tarski, Mostowski *et al.* 1953]). Comparing Gödel's Completeness Theorem with his Incompleteness Theorem, we come to the following conclusion.

Every consistent theory can be extended to a $(\neg, \exists)-$*complete theory* (Gödel's Completeness Theorem) *and this completion preserves decidability, i.e., every consistent and decidable theory can be extended to a consistent, decidable and* $(\neg, \exists)-$*complete theory. But this completion cannot be necessarily effective; i.e., there are some consistent* RE *theories whose all consistent completions are non-*RE (Gödel's Incompleteness Theorem).

So, calling the completeness theorem of Gödel *Completability Theorem* we can call (the first) incompleteness theorem of Gödel (and Rosser) RE-*Incompletabiliy Theorem.*

3.1 An undecidable but RE-completable theory

In this paper we introduce an incomplete but RE-completable theory ($\mathcal{T}$) and a novel RE-incompletable theory ($\mathcal{S}$), and for that we consider the theory of zero, successor and order in the set of natural numbers, i.e., the structure $\langle\mathbb{N}, 0, s, <\rangle$ in which 0 is a constant symbol, s is a unary function symbol and $<$ is a binary relation symbol (interpreted as the zero element, the successor function and the order relation, respectively). This theory is known to be decidable [Enderton 2001], and in fact can be finitely axiomatized as follows

$A_1 : \forall x \forall y (x < y \rightarrow y \not< x)$,

$A_2 : \forall x \forall y \forall z (x < y \wedge y < z \rightarrow x < z)$,

$A_3 : \forall x \forall y (x < y \vee x = y \vee y < x)$,

$A_4 : \forall x \forall y (x < y \longleftrightarrow s(x) < y \vee s(x) = y)$,

$A_5 : \forall x (x \not< 0)$,

$A_6 : \forall x \big(0 < x \rightarrow \exists v (x = s(v))\big)$.

The axioms A_1, A_2, A_3 state that $<$ is a (linear and transitive and antisymmetric, thus a) total ordering, A_4 states that every element has a successor (the successor $s(x)$ of x satisfies $\forall y \big(x < y \leftrightarrow s(x) < y \vee s(x) = y\big)$), A_5 states that there exists a least element (namely 0) and finally A_6 states that every non-zero element has a predecessor. One other advantage of the language $\{0, s, <\}$ is that we have terms for every natural number $n \in \mathbb{N}$:

$$\underline{n} \text{ is the } \{0, s\}\text{–term } s^n 0 = \underbrace{s \cdots s}_{n-\text{times}} 0.$$

To the language $\{0, s, <\}$ we add a ternary relation symbol τ interpreted as:

for $e, x, t \in \mathbb{N}$ the relation $\tau(e, x, t)$ holds when
e is a code for a single-input program which halts on input x by time t.

Timing of a program can be measured either by the number of steps that the program runs or just by the conventional seconds, minutes, hours, etc., and programs (say in a fixed programming language like **C++**) can be coded by natural numbers as follows (for example): any such program is a (long) string of ASCII codes, and every ASCII code can be thought of as 8 symbols of 0's and 1's (so, there are 256 ASCII codes). So, any program is a string of 0's and 1's (whose length is a multiple of 8). The set of 0,1-strings can be coded by natural numbers in the following way:

λ	0	1	00	01	10	11	000	001	010	011	100	$\cdots$
0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$

This coding works as follows: given a string of 0's and 1's (take for example 0110), put a 1 at the beginning of it (in our example 10110) and compute its binary value (in our example $2{+}2^2{+}2^4{=}22$) and subtract 1 from it (in our example 21) to get the natural number which is the code of the original string. Conversely, given a natural number (for example 29) find the binary representation of its successor (in our example $30=2+2^2+2^3+2^4=(11110)_2$) and remove the 1 from its beginning (in our example 1110) to get the 0,1-string which corresponds to the given natural number.

Whence, any program can be coded by a natural number constructively, and if a natural number is a code for a program, then that program can be decoded from that number algorithmically. Let us note the ternary relation τ resembles Kleene's T Predicate (see [Kleene 1936]).

Definition 7 (The Theory $\mathcal{T}$)**.** Theory $\mathcal{T}$ is axiomatized by A_1, A_2, A_3, A_4, A_5 and A_6 with the following set of sentences in the language $\{0, s, <, \tau\}$:

$$A_7 : \{\tau(\underline{e}, \underline{x}, \underline{t}) \mid e, x, t \in \mathbb{N} \ \& \ \mathbb{N} \models \tau(e, x, t)\} \qquad \diamond$$

The set of axioms A_7 consists of the sentences $\tau(\underline{e}, \underline{x}, \underline{t})$ (recall that $\underline{n}$ is the $\{0, s\}$−term representing the number $n \in \mathbb{N}$) such that $\tau(e, x, t)$ holds in reality (the single-input program with code e halts on input x by time t).

Remark 8 (RE-completability of $\mathcal{T}$)**.** *The set of sentences in $\mathcal{T}$ is decidable (given any n, m, k one can decide whether $\tau(n, m, k)$ holds or not), and thus $\mathcal{T}$ is an* RE *theory. It is also* RE*-completable, since its extension by the sentence $\forall x \forall y \forall z \big(\tau(x, y, z)\big)$ is a decidable theory (equivalent to the theory of the structure $\langle \mathbb{N}, 0, s, < \rangle$ which is decidable—see [Enderton 2001]).* $\diamond$

The theory $\mathcal{T}$ is undecidable, since the halting problem is undecidable (see e.g., [Epstein & Carnielli 2008]): for any (single-input program with code) $e \in \mathbb{N}$ and any (input) $m \in \mathbb{N}$, let $\varphi_{e,m}$ be the sentence $\exists z \, \tau(\underline{e}, \underline{m}, z)$. Then

$$\begin{aligned} \mathcal{T} \vdash \varphi_{e,m} \quad &\Longleftrightarrow \quad \mathbb{N} \models \tau(\underline{e}, \underline{m}, \underline{t}) \text{ for some } t \\ &\Longleftrightarrow \quad \text{the program } e \text{ eventually halts on input } m. \end{aligned}$$

This can be shown directly, by incorporating the proof of the undecidability of the halting problem.

Theorem 9 (Undecidability of $\mathcal{T}$)**.** *The theory $\mathcal{T}$ is undecidable.*

Proof. If the set $\mathrm{Der}(\mathcal{T})$ is decidable, then so is the set

$$\mathcal{D} = \{n \in \mathbb{N} \mid \mathcal{T} \nvdash \exists z \, \tau(\underline{n}, \underline{n}, z)\}.$$

Whence, there exists a program which on input $n \in \mathbb{N}$ halts whenever $n \in \mathcal{D}$ (and when $n \notin \mathcal{D}$ then the program does not halt and loops forever). Let e be a code for this (single-input) program. Then

the program (with code) e halts on input e $\Longleftrightarrow$ $\mathbb{N} \models \tau(\underline{e}, \underline{e}, \underline{k})$ for some k $\Longleftrightarrow$ $\mathcal{T} \vdash \exists z \, \tau(\underline{e}, \underline{e}, z)$ $\Longleftrightarrow$ $e \notin \mathcal{D}$ $\Longleftrightarrow$ (by e's definition) the program (with code) e does not halt on input e. Contradiction! □

Remark 10. *Thus far, we have shown that an undecidable theory need not be* RE*-incompletable ($\mathcal{T}$). One can also show that an incomplete theory need not be undecidable; to see this consider the theory $\{\exists x \exists y \forall z (z = x \vee z = y)\}$*

in the language of equality ($=$). *This theory is decidable (holds in models of at most two elements) but not complete, since can derive neither* $\forall x\forall y(x=y)$ *nor* $\neg\forall x\forall y(x=y)$. ◇

Corollary 11 (Undecidability of Consistency). *It is not decidable whether a given* RE *theory is consistent or not.*

Proof. By $[T\vdash\varphi]\iff[T\cup\{\neg\varphi\}$ is inconsistent$]$ if consistency of RE theories was decidable then every RE theory would be decidable too. □

Thus, we have shown the existence of an RE theory ($\mathcal{T}$) which is undecidable but RE-completable. Next, we show the existence of an RE theory which is not RE-completable. Before that let us note that the above proof works for any (consistent and RE) theory $T\supseteq\mathcal{T}$ which is sound (i.e., $\mathbb{N}\models T$).

Theorem 12. *There exists no complete, sound and* RE *theory extending* $\mathcal{T}$. *In other words, the theory* $\mathcal{T}$ *cannot be soundly* RE*-completed.* ❑

This is essentially the semantic form of Gödel's first incompleteness theorem. As a corollary we have that the theory of $\langle\mathbb{N},0,s,<,\tau\rangle$ is not RE (nor decidable). Let us note that the above proof of the first (semantic) incompleteness theorem of Gödel is some rephrasing of Kleene's proof (see [Kleene 1936]). For a (single-input) program (with code) e let W_e be the set of the inputs such that e eventually halts on them, i.e., $W_e=\{n\mid\mathbb{N}\models\exists z\,\tau(e,n,z)\}$. By Turing's results it is known that the set $K=\{n\mid n\in W_n\}$ is RE but not decidable (see e.g., [Epstein & Carnielli 2008]). Indeed, its complement $\overline{K}=\{n\mid n\notin W_n\}$ is not RE because every RE set is of the form W_m for some m (see e.g., [Epstein & Carnielli 2008]) and for any n we have $n\in(\overline{K}\setminus W_n)\cup(W_n\setminus\overline{K})$. On the other hand for any RE theory T the set

$$\overline{K}_T=\{n\mid T\vdash\text{``}n\in\overline{K}\text{''}\}=\{n\mid T\vdash\neg\exists z\,\tau(\underline{n},\underline{n},z)\}$$

is RE. Now, if T is sound ($\mathbb{N}\models T$) then $\overline{K}_T\subseteq\overline{K}$. The inclusion must be proper because one of them ($\overline{K}_T$) is RE and the other one ($\overline{K}$) is not RE. If $\overline{K}_T=W_m$ (for some $m\in\mathbb{N}$) then $m\in\overline{K}-\overline{K}_T$: because if $m\in\overline{K}_T(=W_m)$ then $m\in W_m$ and so $m\notin\overline{K}$, and this contradicts the inclusion $\overline{K}_T\subseteq\overline{K}$; thus $m\notin\overline{K}_T$ and so $m\notin W_m$ which implies that $m\in\overline{K}$. Hence, the sentence $\neg\exists z\,\tau(\underline{m},\underline{m},z)$ is true but unprovable in T; thus T is incomplete.

3.2 An RE-incompletable theory

In the above arguments we used the soundness assumption of T (and $\mathcal{T}$). Below, we will introduce a consistent and RE theory $\mathcal{S}$ which is not RE-completable. Let π be a binary function symbol (representing some pairing—for example $\pi(n,m)=(n+m)^2+n$) whose interpretation in $\mathbb{N}$ satisfies the pairing condition: for any $a,b,a',b'\in\mathbb{N}$ we have $\mathbb{N}\models\pi(a,b)=\pi(a',b')$ if and only if $a=a'$ and $b=b'$.

Definition 13 (The Theory $\mathcal{S}$)**.** The theory $\mathcal{S}$ is the extension of the theory $\mathcal{T}$ by the following sets of sentences in the language $\{0, s, <, \tau, \pi\}$:

$A_8 : \{\neg\tau(\underline{e}, \underline{x}, \underline{t}) \mid e, x, t \in \mathbb{N} \ \& \ \mathbb{N} \not\models \tau(e, x, t)\}$

$A_9 : \{\forall x\big(x < \underline{k} \longleftrightarrow \bigvee_{i<k} x = \underline{i}\big) \mid k \in \mathbb{N}\}$ ◇

We now show that the theory $\mathcal{S}$ is not RE-completetable. Let us note that Gödel's original first incompleteness theorem showed the existence of some theory which was not *soundly* RE-completetable. Actually, Gödel used a syntactic notion weaker than "soundness", namely $\omega-$*consistency*, which is stronger than consistency itself. Nowadays it is known that Gödel's proof works for an even weaker condition than ω−consistency, the so called *1-consistency* (see [Isaacson 2011]). It was then Rosser who showed that Gödel's theorem can be proved without using the ω−consistency (even 1-consistency) assumption (see [Rosser 1936] or e.g., [Boolos, Burgess *et al.* 2007]); so the theorem of Gödel-Rosser states the existence of a consistent and RE theory which is not RE-completetable.

Theorem 14 (RE-incompletability of $\mathcal{S}$)**.** *If T is a consistent* RE *theory that extends $\mathcal{S}$ (i.e., $T \supseteq \mathcal{S}$), then T is not complete.*

Proof. Suppose T is a consistent and RE extension of $\mathcal{S}$. We show that T is not complete. For any $a, b \in \mathbb{N}$ let $\varphi_{a,b}$ be the sentence

$$\exists x\big(\tau(\underline{a}, \pi(\underline{a}, \underline{b}), x) \wedge \forall y{<}x\, \neg\tau(\underline{b}, \pi(\underline{a}, \underline{b}), y)\big).$$

Let m be a code of a program which on input $p \in \mathbb{N}$ halts if and only if there are some $k, l \in \mathbb{N}$ such that $p = \pi(k, l)$ (in which case the numbers k and l are unique) and there exists a proof of $\varphi_{k,l}$ in T (i.e., $T \vdash \varphi_{k,l}$). So, if (i) p is not in the range of the function π, or (ii) there are (unique) k, l such that $p = \pi(k, l)$ and $T \nvdash \varphi_{k,l}$, then the program does not halt on p. Whence, the program with code m searches for a proof of $\varphi_{k,l}$ in T on input $\pi(k, l)$.

Also, let n be a code for a program which for an input $p \in \mathbb{N}$ halts if and only if there are some (unique) $k, l \in \mathbb{N}$ such that $p = \pi(k, l)$ and there exists a proof of $\neg\varphi_{k,l}$ in T (i.e., $T \vdash \neg\varphi_{k,l}$). So, if p is not in the range of the function π or if there are (unique) k, l such that $p = \pi(k, l)$ and $T \nvdash \neg\varphi_{k,l}$ then the program with code n does not halt on p. Again, this program searches for a proof of $\neg\varphi_{k,l}$ in T on input $\pi(k, l)$.

We prove that $\varphi_{n,m}$ is independent from T, i.e., $T \nvdash \varphi_{n,m}$ and $T \nvdash \neg\varphi_{n,m}$.

(1) If $T \vdash \varphi_{n,m}$ then by the consistency of T we have $T \nvdash \neg\varphi_{n,m}$. So, on input $\pi(n, m)$ the program with code m halts and the program with code n does not halt. Whence, for some natural number t, $\mathbb{N} \models \tau(m, \pi(n, m), t)$ and for every natural number s, $\mathbb{N} \not\models \tau(n, \pi(n, m), s)$. So by A_7 and A_8 for that (fixed) $t \in \mathbb{N}$ we have $T \vdash \tau(\underline{m}, \pi(\underline{n}, \underline{m}), \underline{t})$, and for every $s \in \mathbb{N}$ we have $T \vdash \neg\tau(\underline{n}, \pi(\underline{n}, \underline{m}), \underline{s})$. Thus, $T \vdash \bigwedge_{i\leqslant t}\neg\tau(\underline{n}, \pi(\underline{n}, \underline{m}), \underline{i})$, and so by A_9, we conclude that $T \vdash \forall x{\leqslant}\underline{t}\, \neg\tau(\underline{n}, \pi(\underline{n}, \underline{m}), x)$, therefore

(i) $T \vdash \forall x\big(\tau(\underline{n}, \pi(\underline{n}, \underline{m}), x) \rightarrow x{>}\underline{t}\big)$.

Also by $T \vdash \tau(\underline{m}, \pi(\underline{n}, \underline{m}), \underline{t})$ we get

(ii) $T \vdash \forall x > \underline{t}\big(\exists y < x\, \tau(\underline{m}, \pi(\underline{n}, \underline{m}), y)\big)$.

Combining these two conclusions we infer that

$$T \vdash \forall x\ \big(\tau(\underline{n}, \pi(\underline{n}, \underline{m}), x) \rightarrow \exists y < x\, \tau(\underline{m}, \pi(\underline{n}, \underline{m}), y)\big).$$

On the other hand by the definition of $\varphi_{a,b}$ we have

$$\varphi_{n,m} \equiv \exists x\big(\tau(\underline{n}, \pi(\underline{n}, \underline{m}), x) \wedge \forall y < x\, \neg\tau(\underline{m}, \pi(\underline{n}, \underline{m}), y)\big),$$

and so $\neg\varphi_{n,m} \equiv \forall x\big(\tau(\underline{n}, \pi(\underline{n}, \underline{m}), x) \rightarrow \exists y < x\, \tau(\underline{m}, \pi(\underline{n}, \underline{m}), y)\big)$.

Thus we deduced $T \vdash \neg\varphi_{n,m}$ from the assumption $T \vdash \varphi_{n,m}$; contradiction! Whence, $T \nvdash \varphi_{n,m}$.

(2) If $T \vdash \neg\varphi_{n,m}$ then (again) by the consistency of T we have $T \nvdash \varphi_{n,m}$. So, on input $\pi(n, m)$ the program with code n halts and the program with code m does not halt. Whence, for some natural number t, $\mathbb{N} \models \tau(n, \pi(n, m), t)$ and for every natural number s, $\mathbb{N} \not\models \tau(m, \pi(n, m), s)$. Similarly to the above we can conclude that $T \vdash \tau(\underline{n}, \pi(\underline{n}, \underline{m}), \underline{t})$ and $T \vdash \forall y < \underline{t}\, \neg\tau(\underline{m}, \pi(\underline{n}, \underline{m}), y)$. Thus (for $x = \underline{t}$) we have

$$T \vdash \exists x\big(\tau(\underline{n}, \pi(\underline{n}, \underline{m}), x) \wedge \forall y < x\, \neg\tau(\underline{m}, \pi(\underline{n}, \underline{m}), y)\big)$$

or $T \vdash \varphi_{n,m}$; contradiction! So, $T \nvdash \neg\varphi_{n,m}$.

Whence, T is not complete. □

The above proof is effective, in the sense that given an RE theory (by a code for a program that generates its elements) that extends $\mathcal{S}$ one can generate (algorithmically) a sentence which is independent from that theory. Let us note that for proving RE-incompletability of theories, it suffices to interpret $\mathcal{S}$ in them. So, the theories Q, R (see [Tarski, Mostowski *et al.* 1953]) and Peano's Arithmetic PA are all RE-incompletable (or, essentially undecidable).

4 Rice's theorem for RE theories

In this last section we show a variant of Rice's Theorem for logical theories. In [Oliveira & Carnielli 2008] the authors (claimed to) had shown that an analogue of Rice's theorem holds for finitely axiomatizable first order theories. Unfortunately, the result was too beautiful to be true [Oliveira & Carnielli 2009] and it turned out that Rice's theorem cannot hold for finite theories. However, we show that this theorem holds for RE theories, a result which is not too different from Rice's original theorem. Recall that two theories T_1 and T_2 are equivalent when they prove the same (and exactly the same) sentences (i.e., $\mathrm{Der}(T_1) = \mathrm{Der}(T_2)$).

Definition 15 (Property of Theories). A property of (first order logical) theories is a set of natural numbers $\mathcal{P} \subseteq \mathbb{N}$ such that for any $m, n \in \mathbb{N}$ if the theory generated by the program with code m is equivalent to the theory generated by the program with code n, then $m \in \mathcal{P} \longleftrightarrow n \in \mathcal{P}$.

So, a theory is said to have the property $\mathcal{P}$ when a code for generating its set belongs to $\mathcal{P}$. A property of theories is a non-trivial property when some theories have that property and some do not. $\Diamond$

Example 16. The followings are some non-trivial properties of RE theories:

- *Universal Axiomatizability*: theories axiomatizable by sentences of the form $\forall x_1 \ldots \forall x_n \theta(x_1, \ldots, x_n)$ for quantifier-free θ's;
- *Finite Axiomatizability*: being equivalent to a finite theory;
- *Decidability* (of the set of the theorems of the theory);
- *Having a Finite Model*;
- *Completeness*;
- *Consistency*. $\Diamond$

Remark 17. *For any non-trivial property $\mathcal{P}$ either (i) no inconsistent theory has the property $\mathcal{P}$ or (ii) all inconsistent theories have the property $\mathcal{P}$. Because when an inconsistent theory breaks into $\mathcal{P}$ then all the other inconsistent theories (being equivalent to each other) come in.* $\Diamond$

Before proving Rice's Theorem let us have a look at (a variant of) Craig's trick. For an RE theory $T = \{T_1, T_2, T_3, \cdots\}$ the proof predicate "p is a proof of φ in T", for given sequence of sentences p and sentence φ, might not be decidable when the set $\{T_1, T_2, T_3, \cdots\}$ is not decidable. Note that "the sequence p is a proof of φ in T" when every element of p is either a (first order) logical axiom (which can be decided) or is an element of T or can be deduced from two previous elements by an inference rule, and the last element of p is φ. Thus decidability of the set T is essential for the decidability of the proof sequences of T. But if we consider the theory $\widehat{T} = \{\widehat{T}_1, \widehat{T}_2, \widehat{T}_3, \cdots\}$ where $\widehat{T}_m = \bigwedge_{i \leqslant m} T_i$ then the set $\widehat{T}$ is decidable, because if $\mathcal{A}$ is an algorithm that outputs (generates) the infinite sequence $\langle T_1, T_2, T_3, \cdots \rangle$ in this order (in case T is finite the sequence is eventually constant), then for any given sentence ψ we can decide if $\psi \in \widehat{T}$ or not by checking if ψ is a conjunction of some sentences $\psi = \psi_1 \wedge \cdots \wedge \psi_m$ (if ψ is not of this form, then already $\psi \notin \widehat{T}$) such that $\mathcal{A}$'s ith output is ψ_i for $i = 1, \ldots, m$ (if not then again $\psi \notin \widehat{T}$). Whence the predicate of being a proof of φ in $\widehat{T}$, i.e., "the sequence p is a proof of sentence φ in $\widehat{T}$", is decidable; moreover the theories T and $\widehat{T}$ are equivalent, and the theory $\widehat{T}$ can be algorithmically constructed from given theory T.

Theorem 18 (Analogue of Rice's Theorem). *All the non-trivial properties of* RE *theories are undecidable.*

Proof. Assume a non-trivial property $\mathcal{P}$ of RE theories is decidable, i.e., there exists an algorithm which on input $n \in \mathbb{N}$ decides whether $n \in \mathcal{P}$ (i.e., whether the theory generated by the program with code n has the property $\mathcal{P}$). Without loss of generality we can assume that no inconsistent theory has the property $\mathcal{P}$ (otherwise take the complement of $\mathcal{P}$). Fix a consistent RE theory, say, $S = \{S_1, S_2, S_3, \cdots\}$ that has the property $\mathcal{P}$ (S could be finite in which case the sequence $\{S_i\}_i$ is eventually constant) and fix a sentence ψ. For any given RE theory T we construct the theory $T' = \{T'_1, T'_2, T'_3, \cdots\}$ as follows: let $T'_k = S_k$ if k is not a (code of a) proof of $\psi \wedge \neg\psi$ in $\widehat{T}$ (see above); otherwise let $T'_k = \psi \wedge \neg\psi$. Note that this construction is algorithmic, since being a (code for a) proof of $\psi \wedge \neg\psi$ in the decidable set $\widehat{T}$ is decidable (and the theory $\widehat{T}$ can be constructed algorithmically for a given RE theory T); moreover, the theory T' is RE. Now, if T is consistent then $T' = S$ has the property $\mathcal{P}$ and if T is not consistent then T' is an inconsistent theory (because then for some k, $T'_k = \psi \wedge \neg\psi$) and so does not have the property $\mathcal{P}$. Whence, for any RE theory T we have the RE theory T' in such a way that

$$\big[T \text{ is consistent}\big] \iff \big[T' \text{ has the property } \mathcal{P}\big].$$

Now by Corollary 11 the property $\mathcal{P}$ is not decidable. □

Finally, we note that as a corollary to the above theorem, finite axiomatizability of RE theories is not a decidable property; and there exists a decidable non-trivial property for finite theories: for a fixed decidable theory (like the theory $\{A_1, \cdots, A_6\}$ in Definition 7), say F, it is decidable whether a given finite theory T is included in F (i.e., if F can prove all the sentences of T).

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Bibliography

BEKLEMISHEV, Lev D. [2010], Gödel incompleteness theorems and the limits of their applicability: I, *Russian Mathematical Surveys*, 65(5), 857–899, doi: 10.1070/RM2010v065n05ABEH004703.

BERTO, Francesco [2009], *There's Something about Gödel: The Complete Guide to the Incompleteness Theorem*, Malden, MA: Wiley-Blackwell.

BOOLOS, George S., BURGESS, John P., & JEFFREY, Richard C. [2007], *Computability and Logic*, Cambridge; New York: Cambridge University Press, 5th edn.

CHISWELL, Ian & HODGES, Wilfrid [2007], *Mathematical Logic*, London; New York: Oxford University Press.

CRAIG, William [1953], On axiomatizability within a system, *The Journal of Symbolic Logic*, 18(1), 30–32, doi:10.2307/2266324.

Enderton, Herbert B. [2001], *A Mathematical Introduction to Logic*, San Diego: Academic Press, 2nd edn.

Epstein, Richard L. & Carnielli, Walter A. [2008], *Computability: Computable Functions, Logic, and the Foundations of Mathematics*, Pacific Grove: Advanced Reasoning Forum, 3rd edn.

Isaacson, Daniel [2011], Necessary and sufficient conditions for undecidability of the Gödel sentence and its truth, in: *Logic, Mathematics, Philosophy, Vintage Enthusiasms: Essays in honour of John L. Bell*, edited by D. DeVidi, M. Hallett, & P. Clark, Dordrecht; New York: Springer, 135–152, doi:10.1007/978-94-007-0214-1_7.

Kaye, Richard W. [2007], *The Mathematics of Logic: A guide to completeness theorems and their applications*, Cambridge; New York: Cambridge University Press.

Kleene, Stephen C. [1936], General recursive functions of natural numbers, *Mathematische Annalen*, 112(1), 727–742.

—— [1950], Symmetric form of Gödel's theorem, *Indagationes Mathematicae*, 12, 244–246.

—— [1952], *Introduction to Metamathematics*, Amsterdam: North-Holland.

Lafitte, Grégory [2009], Busy beavers gone wild, in: *Proceedings of the International Workshop on the Complexity of Simple Programs*, edited by T. Neary, D. Woods, T. Seda, & N. Murphy, Cork, Ireland: Open Publishing Association, Electronic Proceedings in Theoretical Computer Science, vol. 1, 123–129, doi:10.4204/EPTCS.1.12.

Li, Ming & Vitányi, Paul M. B. [2008], *An Introduction to Kolmogorov Complexity and Its Applications*, New York: Springer, 3rd edn.

Oliveira, Igor Carboni & Carnielli, Walter [2008], The Ricean objection: An analogue of Rice's theorem for first–order theories, *Logic Journal of the IGPL*, 16(6), 585–590, doi:10.1093/jigpal/jzn023.

—— [2009], Erratum to "The Ricean objection: An analogue of Rice's theorem for first–order theories", *Logic Journal of the IGPL*, 17(6), 803–804, doi: 10.1093/jigpal/jzp030.

Rosser, Barkley [1936], Extensions of some theorems of Gödel and Church, *The Journal of Symbolic Logic*, 1(3), 87–91, doi:10.2307/2269028.

Tarski, Alfred, Mostowski, Andrzej, & Robinson, Raphael M. [1953], *Undecidable Theories*, Amsterdam: North-Holland, reprinted by Dover Publications 2010.

van Dalen, Dirk [2013], *Logic and Structure*, London; New York: Springer, 5th edn.

Wasserman, Wayne U. [2008], It is "Pulling a rabbit out of the hat": typical diagonal lemma "proofs" beg the question, doi:10.2139/ssrn.1129038.