Provably total functions of Basic Arithmetic

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It is shown that all the provably total functions of Basic Arithmetic BA, a theory introduced by Ruitenburg based on Predicate Basic Calculus, are primitive recursive. Along the proof a new kind of primitive recursive realizability to which BA is sound, is introduced. This realizability is similar to Kleene's recursive realizability, except that recursive functions are restricted to primitive recursives.

1 Introduction

Basic Propositional Logic was first introduced by Visser [26][1)] and was extended to Basic Predicate Calculus by Ruitenburg [17]. "It is the sub-logic of the intuitionistic logic which is characterized by the class of Kripke frames with transitive (but not necessarily reflexive) accessibility relations, so that the modal logic K4 corresponds to this logic by Gödel translation of the intuitionistic logic into the modal logic S4. It has the peculiarity on modus ponens that, although $\vdash A$ and $\vdash A \supset B$ always imply $\vdash B$, yet $\Gamma \vdash A$ and $\Gamma \vdash A \supset B$ do not necessarily imply $\Gamma \vdash B$" ([24]). For the chronology of the advancements on this rather new logic the reader is referred to [1 – 6, 15 – 18, 22]. Although in [17] Ruitenburg "hope(s) that one day a better name (for Basic Logic) comes along" the term "Basic Logic" is also used for naming another completely different non-classical logic, see e. g. [21] and its references.

Basic Arithmetic (BA) was introduced by Ruitenburg as a counterpart of Heyting Arithmetic HA (on intuitionistic logic) and of Peano Arithmetic PA (on classical logic) based on basic logic. In Section 2 we present an axiomatization of BA given in [17].

In this paper we prove that every provably total function of BA is primitive recursive (see Corollary 4.5). This theorem shows the constructive feature of BA (as expected) and its weakness in comparison to HA and to PA (see Remark 4.6).

One of the well-known tools for characterizing provably total functions of a constructive theory is *realizability*. Recursive realizability was introduced by Kleene (see [11] and [25] for the definitions and their history). A corollary of the soundness of HA with respect to Kleene's recursive realizability is a specification of provably total functions of HA: "they are all recursive". This result has been refined by Damnjanovic [7] using a special realizability, called $<\varepsilon_0$-recursive realizability, in which recursive functions in Kleene's realizability are limited to $<\varepsilon_0$-recursive functions. The improved result is: "provably total functions of HA are precisely $<\varepsilon_0$-recursive functions" (in [20] a simpler proof of this statement is given, see also [29, p. 43]).

Wehmeier [29] used Kleene's realizability to characterize provably total functions of $i\,\Sigma_1$: "each of them is primitive recursive". This had (already) been proven by Damnjanovic [7] using so-called strictly primitive recursive realizability (see also [9]).

In this paper we introduce a kind of primitive recursive realizability in which the recursive functions used by Kleene are limited to primitive recursives.

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1) It is sometimes called "Visser's Propositional Logic", see e. g. [22].

It is well-know that there is no (primitive recursive) enumeration of all primitive recursive functions. Damnjanovic [7, 9] uses Grzegorczyk hierarchy to overcome this obstacle and his definition of realizability is very different from Kleene's (as he calls it "non-classical"). We deal with this problem by using a formula $\mathrm{PR}(x)$ that *defines* the functions in the program of which no minimization is used. Hence if $\mathrm{PR}(x)$ holds, then the function whose program has the (Gödel) code x is primitive recursive, but not (necessarily) the other way round (see Section 3).

Another kind of realizability by primitive recursive functions was proposed by López-Escobar [13], so called "prim-realizability", which is close (but not identical) to ours. His enumeration of primitive recursive functions makes use of Kleene's predicate $\mathrm{In}^m(b)$, which is true when "b is an index for determining a function of $(b)_1$ arguments from any function ψ of m arguments by adjoining instances of primitive recursive schemata to the true numerical equations for ψ" ([13, Section 4], see also [12] and [10]). Plisko [14] uses "definable" recursive functions in his Σ_n-realizability, such that that the recursive functions replaced for general recursive functions in Kleene's realizability are definable by Σ_n-formulas. Viter's dissertation [28] (see also [27]) presents some further investigations made on this primitive recursive realizability introduced by the author, for the first time, in [19].

2 Basic Arithmetic

Basic Arithmetic is the basic logic equivalent of Heyting Arithmetic over intuitionistic logic, and of Peano Arithmetic over classical logic. The non-logical symbols are a constant '0', a unary function symbol 'S' for successor, and the binary function symbols '$\cdot$' and '$+$' (see [17, Section 6]).

The language of Basic Predicate Calculus contains two logical constants $\bot$ (falsehood) and $\top$ (truth) and the logical connectives $\wedge$, $\vee$, $\exists$ and $\forall$. Terms, atomic formulas, and formulas are defined as usual, except that for universal quantification we have the more elaborate rule: if A and B are formulas and $\boldsymbol{x}$ is a finite (possibly empty) sequence of variables, then $\forall\boldsymbol{x}(A\to B)$ is also a formula. Free variables are defined in the obvious way. We may write $A\to B$ for $\forall(A\to B)$, that is, implication is universal quantification with an empty sequence of variables. Given a sequence of variables $\boldsymbol{x}$ without repetitions, we write $A^{\boldsymbol{x}}_{\boldsymbol{t}}$ for the formulas that result from substituting the terms of $\boldsymbol{t}$ for all free occurrences of the variables of $\boldsymbol{x}$ in the formula A (see [17, Section 2]).

The axioms of Basic Arithmetic BA (over the sequent calculus) are:

Ax1 $A\Rightarrow A$,

Ax2 $A\Rightarrow\top$,

Ax3 $\bot\Rightarrow A$,

Ax4 $A \wedge \exists xB\Rightarrow\exists x\,(A \wedge B)$, in which x is not free in A,

Ax5 $A \wedge (B \vee C)\Rightarrow(A \wedge B) \vee (A \wedge C)$,

Ax6 $\forall\boldsymbol{x}\,(A\to B) \wedge \forall\boldsymbol{x}\,(B\to C)\Rightarrow\forall\boldsymbol{x}\,(A\to C)$,

Ax7 $\forall\boldsymbol{x}\,(A\to B) \wedge \forall\boldsymbol{x}\,(A\to C)\Rightarrow\forall\boldsymbol{x}\,(A\to B \wedge C)$,

Ax8 $\forall\boldsymbol{x}\,(B\to A) \wedge \forall\boldsymbol{x}\,(C\to A)\Rightarrow\forall\boldsymbol{x}\,(B \vee C\to A)$,

Ax9 $\forall\boldsymbol{x}\,(A\to B)\Rightarrow\forall\boldsymbol{x}\,(A^{\boldsymbol{x}}_{\boldsymbol{t}}\to B^{\boldsymbol{x}}_{\boldsymbol{t}})$, in which no variable in $\boldsymbol{t}$ is bounded by a quantifier of A or B,

Ax10 $\forall\boldsymbol{x}\,(A\to B)\Rightarrow\forall\boldsymbol{y}\,(A\to B)$, in which no variable in $\boldsymbol{y}$ is free in the left hand side,

Ax11 $\forall\boldsymbol{y}x\,(B\to A)\Rightarrow\forall\boldsymbol{y}\,(\exists x\,B\to A)$, in which x is not free in A,

Ax12 $\Rightarrow x = x$,

Ax13 $x = y \wedge A\Rightarrow A^x_y$, for atomic A,

Ax14 $S(x) = S(y)\Rightarrow x = y$,

Ax15 $S(x) = 0\Rightarrow\bot$,

Ax16 $\Rightarrow x + 0 = x$,

Ax17 $\Rightarrow x \cdot 0 = 0$,

Ax18 $\Rightarrow x + S(y) = S(x + y)$,

Ax19 $\Rightarrow x \cdot S(y) = (x \cdot y) + x$,

Ax20 $\forall x\boldsymbol{y}\,(A\to A^x_{Sx})\Rightarrow\forall x\boldsymbol{y}\,(A^x_0\to A)$.

The rules of BA are:

Ru1 $\dfrac{A \Rightarrow B \quad B \Rightarrow C}{A \Rightarrow C}$,

Ru2 $\dfrac{A \Rightarrow B \quad A \Rightarrow C}{A \Rightarrow B \wedge C}$,

Ru3 $\dfrac{A \Rightarrow B \wedge C}{A \Rightarrow B}$, $\dfrac{A \Rightarrow B \wedge C}{A \Rightarrow C}$,

Ru4 $\dfrac{B \Rightarrow A \quad C \Rightarrow A}{B \vee C \Rightarrow A}$,

Ru5 $\dfrac{B \vee C \Rightarrow A}{B \Rightarrow A}$, $\dfrac{B \vee C \Rightarrow A}{C \Rightarrow A}$,

Ru6 $\dfrac{A \Rightarrow B}{A^{\boldsymbol{x}}_{\boldsymbol{t}} \Rightarrow B^{\boldsymbol{x}}_{\boldsymbol{t}}}$, in which no variable in $\boldsymbol{t}$ is bounded in A or B,

Ru7 $\dfrac{B \Rightarrow A}{\exists x\, B \Rightarrow A}$, in which x is not free in A,

Ru8 $\dfrac{\exists x\, B \Rightarrow A}{B \Rightarrow A}$, in which x is not free in A,

Ru9 $\dfrac{A \wedge B \Rightarrow C}{A \Rightarrow \forall \boldsymbol{x}\, (B \to C)}$, in which no variable in $\boldsymbol{x}$ is free in A,

Ru10 $\dfrac{A \Rightarrow A^x_{Sx}}{A^x_0 \Rightarrow A}$

3 Primitive recursive realizability

Unary recursive functions can be "enumerated" recursively (see e. g. [23]). Let φ_x be the (unique) unary recursive function whose program has the (Gödel) code x (as in e. g. [23], in some literature it is denoted by $\{x\}$). Take $\langle\, .\, ,\, .\, \rangle$ be a fixed pairing function (e. g. $\langle x,y\rangle = \frac{1}{2}(x+y)(x+y+1)+y$) with projections π_1 and π_2 ($\pi_1(\langle x,y\rangle) = x$, $\pi_2(\langle x,y\rangle) = y$). For a sequence $\boldsymbol{x} = \langle x_1, x_2, \ldots , x_m\rangle$, $\varphi_a(\boldsymbol{x})$ is understood as $\varphi_a(\langle x_1, \langle x_2 ... \langle x_{m-1}, x_m\rangle\rangle\rangle)$. We note that any statement involving $\varphi_a(x)$ can be written in the language of arithmetic: a proposition like $\mathcal{P}(\varphi_a(x))$ is $\exists z\, \big(\mathrm{T}(a,x,z) \wedge \mathcal{P}(\mathrm{U}(z))\big)$, where T is Kleene's T-predicate and U is result-extracting function (see e. g. [25]).

Throughout, we take the language of $\mathbb{N}$ to contain function symbols for all primitive recursive functions.

Take $\mathrm{PR}(x)$ be the formula expressing that "in the program[2] x there is no use of minimization". So if $\mathbb{N} \vDash \mathrm{PR}(x)$, then φ_x is primitive recursive but not the vice versa: let m be the code of the program which gives out "$\mu y\, (x \leq 2y)$" (as output) for the input "x", where μ is the minimization operation. It is not difficult to see that φ_m is primitive recursive (that is $\varphi_m(x) = \lceil x/2 \rceil$), but $\mathbb{N} \nvDash \mathrm{PR}(m)$.

However for every primitive recursive function f there is a natural number n such that $\varphi_n = f$ and $\mathrm{PR}(n)$ is true.

Definition 3.1 $x\, \mathbf{r}^{\mathrm{PR}} A$ is defined by induction on the complexity of A:

- $x\, \mathbf{r}^{\mathrm{PR}} p \equiv p$, for atomic p, and $p = \top, \bot$.
- $x\, \mathbf{r}^{\mathrm{PR}} A \wedge B \equiv \big(\pi_1(x)\, \mathbf{r}^{\mathrm{PR}} A\big) \wedge \big(\pi_2(x)\, \mathbf{r}^{\mathrm{PR}} B\big)$.
- $x\, \mathbf{r}^{\mathrm{PR}} A \vee B \equiv \big(\pi_1(x) = 0 \wedge \pi_2(x)\, \mathbf{r}^{\mathrm{PR}} A\big) \vee \big(\pi_1(x) \neq 0 \wedge \pi_2(x)\, \mathbf{r}^{\mathrm{PR}} B\big)$.
- $x\, \mathbf{r}^{\mathrm{PR}} \exists z\, A(z) \equiv \pi_2(x)\, \mathbf{r}^{\mathrm{PR}} A(\pi_1(x))$.
- $x\, \mathbf{r}^{\mathrm{PR}} \forall \boldsymbol{z}\, (A(\boldsymbol{z}) \to B(\boldsymbol{z})) \equiv \mathrm{PR}(x) \wedge \forall y, \boldsymbol{z} \big(y\, \mathbf{r}^{\mathrm{PR}} A(\boldsymbol{z}) \to \varphi_x(y, \boldsymbol{z})\, \mathbf{r}^{\mathrm{PR}} B(\boldsymbol{z})\big)$.

We extend this realizability to sequents involving the free variables of its formulas:

Definition 3.2 Let $\boldsymbol{z} = (z_1, \ldots , z_m)$ be the sequence of all free variables in the sequent $A \Rightarrow B$ in the appearing order. Then $x\, \mathbf{r}^{\mathrm{PR}} (A \Rightarrow B)$ is defined by $\mathrm{PR}(x) \wedge \forall y, \boldsymbol{z}\, \big(y\, \mathbf{r}^{\mathrm{PR}} A(\boldsymbol{z}) \to \varphi_x(y, \boldsymbol{z})\, \mathbf{r}^{\mathrm{PR}} B(\boldsymbol{z})\big)$.

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[2] Program of a recursive function shows how it is defined in terms of the "Zero", "Successor" and "Projection" functions, and "Composition", "Primitive Recursion" and "Minimization" of the functions. Note that, by choosing a suitable coding, we can assume that every natural number is a program's code.

We say *the primitive recursive function* f *realizes* $A \Rightarrow B$, if for some natural number n, $\mathrm{PR}(n)$, $\varphi_n = f$, and $n\,\mathbf{r}^{\mathrm{PR}}(A \Rightarrow B)$ hold. Throughout the paper 'realizability' and 'primitive recursive realizability' are used interchangeably when there is no ambiguity.

Remark 3.3 In Definition 3.2 we note that the sequence $\boldsymbol{z}$ might be empty, in that case the definition will be $\mathrm{PR}(x) \wedge \forall y\,\bigl(y\,\mathbf{r}^{\mathrm{PR}} A \to \varphi_x(y)\,\mathbf{r}^{\mathrm{PR}} B\bigr)$.

Remark 3.4 Considering the free variables of the sequent in Definition 3.2 may seem odd at the first glance. The only reason for that is simply Corollary 4.5 below. We could define $x\,\mathbf{r}^{\mathrm{PR}}(A \Rightarrow B)$ as straightforward as $\mathrm{PR}(x) \wedge \forall y\,\bigl(y\,\mathbf{r}^{\mathrm{PR}} A \to \varphi_x(y)\,\mathbf{r}^{\mathrm{PR}} B\bigr)$, but then only the part (i) of Corollary 4.5 could be proven, not (ii).

Recall S-m-n Theorem (see e. g. [23]): For every m, $n \geq 1$ and z, there is a primitive recursive function S^m_n such that for every m-tuple $\boldsymbol{x}$ and n-tuple $\boldsymbol{y}$, $\varphi_{S^m_n(\boldsymbol{x})}(\boldsymbol{y}) = \varphi_z(\boldsymbol{x}, \boldsymbol{y})$.

Our main theorem is the soundness of $\mathbf{r}^{\mathrm{PR}}$ with respect to BA:

Theorem 3.5 *For all sequents* $A \Rightarrow B$*, if* $\mathsf{BA} \vdash A \Rightarrow B$*, then* $\mathbb{N} \models n\,\mathbf{r}^{\mathrm{PR}}(A \Rightarrow B)$ *for some natural* n*.*

Proof. By induction on the length of the proof of the sequent: we show that for each axiom there is a natural number realizing it (in $\mathbb{N}$) and for any realizer of the hypothesis of the rules there is a natural number realizing its conclusion. For simplicity in all cases we take $\boldsymbol{z}$ to be all the non-presented free variables of A, B and C.

Axioms. For realizing a sequent like $A \Rightarrow B$, it is enough to find a *primitive recursive* function f with the property that for every m, $f(m, \boldsymbol{z})\,\mathbf{r}^{\mathrm{PR}} B$ if $m\,\mathbf{r}^{\mathrm{PR}} A$.

For Ax1, Ax2, and Ax3 let $f(u, \boldsymbol{z}) = u$.

For Ax12, Ax15, Ax16, and Ax17 let $f(u, x) = 0$.

For Ax14, Ax18, and Ax19 let $f(u, x, y) = u$.

For Ax4 and Ax5 let $f(u, \boldsymbol{z}) = \langle \pi_1 \pi_2(u), \langle \pi_1(u), \pi_2 \pi_2(u) \rangle \rangle$.

For Ax6 we note that the function g defined by $g(u, \boldsymbol{z}, v, \boldsymbol{x}) = \varphi_{\pi_2(u)}(\varphi_{\pi_1(u)}(v, \boldsymbol{x}), \boldsymbol{x})$ is recursive, now let f be a primitive recursive function such that $\varphi_{f(u,\boldsymbol{z})}(v, \boldsymbol{x}) = g(u, \boldsymbol{z}, v, \boldsymbol{x})$ (such an f exists by S-m-n Theorem).

For Ax7, similar to the previous case, take a primitive recursive function f such that

$$\varphi_{f(u,\boldsymbol{z})}(v, \boldsymbol{x}) = \langle \varphi_{\pi_1(u)}(v, \boldsymbol{x}), \varphi_{\pi_2(u)}(a, \boldsymbol{x}) \rangle.$$

For Ax8, again similar to the above cases, take a primitive recursive function f with the property

$$\varphi_{f(u,\boldsymbol{z})}(v, \boldsymbol{x}) = \begin{cases} \varphi_{\pi_1(u)}(\pi_2(v), \boldsymbol{x}) & \text{if } \pi_1(v) = 0, \\ \varphi_{\pi_2(u)}(\pi_2(v), \boldsymbol{x}) & \text{if } \pi_1(v) \neq 0. \end{cases}$$

For Ax9, without loss of generality we may assume $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y})$, and that all free variables occurring in $\boldsymbol{t}$ are in $\boldsymbol{x}$. Then take a primitive recursive function f satisfying $\varphi_{f(u,\boldsymbol{y})}(v, \boldsymbol{x}) = \varphi_u(v, \boldsymbol{x})$.

For Ax10, we can assume $\boldsymbol{x}$, $\boldsymbol{y} \subseteq \boldsymbol{z}$. Take a primitive recursive function f satisfying $\varphi_{f(u,\boldsymbol{z})}(\boldsymbol{y}) = \varphi_u(\boldsymbol{x})$.

For Ax11 assume $\boldsymbol{y}\,x\,\boldsymbol{z}$ are all the variables ($\boldsymbol{z}$ being free) of the sequent in the appearing order, and take a primitive recursive function f satisfying $\varphi_{f(u,\boldsymbol{z})}(v, \boldsymbol{y}) = \varphi_u(\pi_2(v), \boldsymbol{y}, \pi_1(v))$.

For Ax13, by taking $x\,y\,\boldsymbol{z}$ to be all variables ($\boldsymbol{z}$ being free) of the sequent in the appearing order, we can put $f(u, x, y, \boldsymbol{z}) = \pi_2(u)$.

Finally for Ax20, assuming that $x\,\boldsymbol{y}\,\boldsymbol{z}$ are all variables of the sequent ($\boldsymbol{z}$ being free) in the appearing order, we can take a primitive recursive function f satisfying

$$\varphi_{f(u,\boldsymbol{z})}(v, 0, \boldsymbol{y}) = v, \quad \varphi_{f(u,\boldsymbol{z})}(v, x+1, \boldsymbol{y}) = \varphi_u(\varphi_{f(u,\boldsymbol{z})}(v, x, \boldsymbol{y}), x, \boldsymbol{y}).$$

For all cases it can be proven that the function f (and g) realizes the correspondent axiom. We verify it for Ax6 and Ax20.

For Ax6, take m be a realizer of $\forall \boldsymbol{x}\,(A(\boldsymbol{x}, \boldsymbol{z}) \to B(\boldsymbol{x}, \boldsymbol{z})) \wedge \forall \boldsymbol{x}\,(B(\boldsymbol{x}, \boldsymbol{z}) \to C(\boldsymbol{x}, \boldsymbol{z}))$. Then

$$\pi_1(m)\,\mathbf{r}^{\mathrm{PR}} \forall \boldsymbol{x}\,(A(\boldsymbol{x}, \boldsymbol{z}) \to B(\boldsymbol{x}, \boldsymbol{z})) \quad \text{and} \quad \pi_2(m)\,\mathbf{r}^{\mathrm{PR}} \forall \boldsymbol{x}\,(B(\boldsymbol{x}, \boldsymbol{z}) \to C(\boldsymbol{x}, \boldsymbol{z})).$$

We show that $f(m, \boldsymbol{z})\,\mathbf{r}^{\mathrm{PR}} \forall \boldsymbol{x}\,(A(\boldsymbol{x}, \boldsymbol{z}) \to C(\boldsymbol{x}, \boldsymbol{z}))$, or, in other words, that

$$\mathrm{PR}(f(m, \boldsymbol{z})) \text{ and } \varphi_{f(m,\boldsymbol{z})}(n, \boldsymbol{x})\,\mathbf{r}^{\mathrm{PR}} C(\boldsymbol{x}, \boldsymbol{z}) \text{ holds for any } n \text{ realizing } A(\boldsymbol{x}, \boldsymbol{z}).$$

Since $\pi_1(m)$ realizes $\forall \boldsymbol{x}\,(A(\boldsymbol{x},\boldsymbol{z}) \to B(\boldsymbol{x},\boldsymbol{z}))$, then $\varphi_{\pi_1(m)}(n,\boldsymbol{x})$ realizes $B(\boldsymbol{x},\boldsymbol{z})$ and similarly, since $\pi_2(m)$ realizes $\forall \boldsymbol{x}\,(A(\boldsymbol{x},\boldsymbol{z}) \to B(\boldsymbol{x},\boldsymbol{z}))$, then $\varphi_{\pi_2(m)}(\varphi_{\pi_1(m)}(n,\boldsymbol{x}),\boldsymbol{x})$ realizes $C(\boldsymbol{x},\boldsymbol{z})$. Finally we note that by the definitionof f, $\varphi_{f(m,\boldsymbol{z})}(n,\boldsymbol{x}) = \varphi_{\pi_2(m)}(\varphi_{\pi_1(m)}(n,\boldsymbol{x}),\boldsymbol{x})$. It remains to show $\mathrm{PR}(f(m,\boldsymbol{z}))$ for any $\boldsymbol{z}$, and this is clear by $\mathrm{PR}(\pi_1(m))$ and $\mathrm{PR}(\pi_2(m))$.

For Ax20, take m be a realizer of $\forall x \boldsymbol{y}\,(A(x) \to A(Sx))$, we show that $f(m,\boldsymbol{z})\,\mathbf{r}^{\mathrm{PR}} \forall x\boldsymbol{y}\,(A(0) \to A(x))$, or, in other words, that $\mathrm{PR}(f(m,\boldsymbol{z}))$ and $\varphi_{f(m,\boldsymbol{z})}(n,x,\boldsymbol{y})\,\mathbf{r}^{\mathrm{PR}} A(x)$ holds for any n which realizes $A(0)$. By the definition of $\varphi_{f(m,\boldsymbol{z})}(n,x,\boldsymbol{y})$ it can be seen that $\mathrm{PR}(f(m,\boldsymbol{z}))$ is true when $\mathrm{PR}(m)$ is so. We show $\varphi_{f(m,\boldsymbol{z})}(n,x,\boldsymbol{y})\,\mathbf{r}^{\mathrm{PR}} A(x)$ by induction on x: For $x = 0$, this is clear, since $\varphi_{f(m,\boldsymbol{z})}(n,0,\boldsymbol{y}) = n$ and $n\,\mathbf{r}^{\mathrm{PR}} A(0)$. For $x+1$, by induction hypothesis $\varphi_{f(m,\boldsymbol{z})}(n,x,\boldsymbol{y})\,\mathbf{r}^{\mathrm{PR}} A(x)$, and since m realizes $\forall x\boldsymbol{y}\,(A(x) \to A(Sx))$, then $\varphi_{f(m,\boldsymbol{z})}(n,x+1,\boldsymbol{y}) = \varphi_m(\varphi_{f(n,\boldsymbol{z})}(m,x,\boldsymbol{y}),x,\boldsymbol{y})\,\mathbf{r}^{\mathrm{PR}} A(Sx)$.

Rules. Similar to the axiom cases, assuming that n (and m) realizes (relize) the hypothesis (hypotheses) of a rule, it is enough to find a primitive recursive function f (primitive recursive functions f and g) which realizes the conclusion(s) of the rule. (Recall that $\boldsymbol{z}$ presents the sequence of all non-shown free variables of A, B and C.)

Ru1: If $n\,\mathbf{r}^{\mathrm{PR}}(A \Rightarrow B)$ and $m\,\mathbf{r}^{\mathrm{PR}}(B \Rightarrow C)$, then the primitive recursive function f defined by

$$f(u,\boldsymbol{z}) = \varphi_m(\varphi_n(u,\boldsymbol{z}),\boldsymbol{z})$$

realizes $A \Rightarrow C$.

Ru2: If $n\,\mathbf{r}^{\mathrm{PR}}(A \Rightarrow B)$ and $m\,\mathbf{r}^{\mathrm{PR}}(A \Rightarrow C)$, then the primitive recursive function f defined by

$$f(u,\boldsymbol{z}) = \langle \varphi_n(u,\boldsymbol{z}), \varphi_m(u,\boldsymbol{z}) \rangle$$

realizes $A \Rightarrow B \wedge C$.

Ru3: If $n\,\mathbf{r}^{\mathrm{PR}}(A \Rightarrow B \wedge C)$, then the primitive recursive functions f and g defined by

$$f(u,\boldsymbol{z}) = \pi_1 \varphi_n(u,\boldsymbol{z}), \qquad g(u,\boldsymbol{z}) = \pi_2 \varphi_n(u,\boldsymbol{z})$$

realize $A \Rightarrow B$ and $A \Rightarrow C$, respectively.

Ru4: If $n\,\mathbf{r}^{\mathrm{PR}}(B \Rightarrow A)$ and $m\,\mathbf{r}^{\mathrm{PR}}(C \Rightarrow A)$, then the primitive recursive function f defined by

$$f(u,\boldsymbol{z}) = \begin{cases} \varphi_n(\pi_2(u),\boldsymbol{z}) & \text{if } \pi_1(u) = 0, \\ \varphi_m(\pi_2(u),\boldsymbol{z}) & \text{if } \pi_1(u) \neq 0. \end{cases}$$

realizes $B \vee C \Rightarrow A$.

Ru5: If $n\,\mathbf{r}^{\mathrm{PR}}(B \vee C \Rightarrow A)$, then the primitive recursive functions f and g defined by

$$f(u,\boldsymbol{z}) = \varphi_n(\langle 0,u \rangle,\boldsymbol{z}), \qquad g(u,\boldsymbol{z}) = \varphi_n(\langle 1,u \rangle,\boldsymbol{z})$$

realize $B \Rightarrow A$ and $C \Rightarrow A$, respectively.

Ru6: If $n\,\mathbf{r}^{\mathrm{PR}}(A \Rightarrow B)$, then $n\,\mathbf{r}^{\mathrm{PR}}(A^{\boldsymbol{x}}_{\boldsymbol{t}} \Rightarrow B^{\boldsymbol{x}}_{\boldsymbol{t}})$.

Ru7: If $n\,\mathbf{r}^{\mathrm{PR}}(B \Rightarrow A)$, assuming that x is free in B and $x\,\boldsymbol{z}$ is the sequence of all free variables of the sequent $B \Rightarrow A$ in the appearing order, f defined by $f(u,\boldsymbol{z}) = \varphi_n(\pi_2(u),\pi_1(u),\boldsymbol{z})$ realizes $\exists x\, B \Rightarrow A$.

Ru8: If $n\,\mathbf{r}^{\mathrm{PR}}(\exists x\, B \Rightarrow A)$ by the above assumption on free variables of $B \Rightarrow A$, then the primitive recursive function f defined by $f(u,x,\boldsymbol{z}) = \varphi_n(\langle x,u \rangle,\boldsymbol{z})$ realizes $B \Rightarrow A$.

Ru9: If $n\,\mathbf{r}^{\mathrm{PR}}(A \wedge B \Rightarrow C)$ and $\boldsymbol{x}\,\boldsymbol{z}$ is the sequence of all free variables of the sequent in the appearing order, then a primitive recursive function f satisfying $\varphi_{f(u,\boldsymbol{z})}(v,\boldsymbol{x}) = \varphi_n(v,\boldsymbol{x},\boldsymbol{z})$ realizes $A \Rightarrow \forall \boldsymbol{x}\,(B \to C)$.

Ru10: If $n\,\mathbf{r}^{\mathrm{PR}}(A \Rightarrow A^x_{Sx})$ and $x\boldsymbol{z}$ is the sequence of the free variables of the sequent $A \Rightarrow A^x_{Sx}$ in the appearing order, then the primitive recursive function f defined by

$$f(u,0,\boldsymbol{z}) = u, \qquad f(u,x+1,\boldsymbol{z}) = \varphi_n(f(u,x,\boldsymbol{z}),x,\boldsymbol{z})$$

realizes $A^x_0 \Rightarrow A$.

It can be shown that if n (and m) realizes (realize) the hypothesis (hypotheses) of the above rules, then the function(s) f (and g) realizes (realize) the conclusion of the rule. We verify this for Ru10.

It is enough to show that f is primitive recursive and $f(m,x,\boldsymbol{z})\,\mathbf{r}^{\mathrm{PR}}A(x)$ for any m realizing $A(0)$. Primitive recursiveness of f is immediate from $\mathrm{PR}(n)$. We show $f(m,x,\boldsymbol{z})\,\mathbf{r}^{\mathrm{PR}}A(x)$ by induction on x: For $x=0$ it is clear, since $f(m,0,\boldsymbol{z})=m$ and $m\,\mathbf{r}^{\mathrm{PR}}A(0)$. For $x+1$, by induction hypothesis $f(m,x,\boldsymbol{z})\,\mathbf{r}^{\mathrm{PR}}A(x)$ and since $n\,\mathbf{r}^{\mathrm{PR}}(A(x)\Rightarrow A(Sx))$, then $\varphi_n(f(m,x,\boldsymbol{z}),x,\boldsymbol{z})\big(=f(m,x+1,\boldsymbol{z})\big)\,\mathbf{r}^{\mathrm{PR}}A(Sx)$. □

Remark 3.6 Heyting Arithmetic HA can be axiomatized by adding the axiom scheme $\top\to A\Rightarrow A$ to BA (see [17, Proposition 4.1]). It should be no surprise that $\top\to A\Rightarrow A$ is *not* primitive recursively realizable for some (non BA-provable) formula A: for simplicity assume A has no free variable, a realizer for $\top\to A\Rightarrow A$ could be like $f(x)=\varphi_x(0)$, since for any realizer x of $\top\to A$, $\varphi_x(0)$ realizes A as 0 realizes the truth (=$\top$), but apparently f is not primitive recursive (it is not even total).

4 Provably total functions of BA

For characterizing the class of provably total functions of a constructive theory **q**-realizability proves to be more convenient (see e. g. [29]).

Primitive recursive **q**-realizability can be defined using the known modifications:

Definition 4.1 $x\,\mathbf{q}^{\mathrm{PR}}A$ is defined by induction on A:

- $x\,\mathbf{q}^{\mathrm{PR}}p \equiv p$ for atomic p and $p=\top,\ \bot$.
- $x\,\mathbf{q}^{\mathrm{PR}}A\wedge B \equiv (\pi_1(x)\,\mathbf{q}^{\mathrm{PR}}A)\wedge(\pi_2(x)\,\mathbf{q}^{\mathrm{PR}}B)$.
- $x\,\mathbf{q}^{\mathrm{PR}}A\vee B \equiv (\pi_1(x)=0\wedge\pi_2(x)\,\mathbf{q}^{\mathrm{PR}}A)\vee(\pi_1(x)\neq 0\wedge\pi_2(x)\,\mathbf{q}^{\mathrm{PR}}B)$.
- $x\,\mathbf{q}^{\mathrm{PR}}\exists z\,A(z) \equiv \pi_2(x)\,\mathbf{q}^{\mathrm{PR}}A(\pi_1(x))$.
- $x\,\mathbf{q}^{\mathrm{PR}}\forall\boldsymbol{z}\,(A(\boldsymbol{z})\to B(\boldsymbol{z})) \equiv \mathrm{PR}(x)\wedge\forall y\,\boldsymbol{z}(y\,\mathbf{q}^{\mathrm{PR}}A(\boldsymbol{z})\to\varphi_n(y,\boldsymbol{z})\,\mathbf{q}^{\mathrm{PR}}B(\boldsymbol{z}))\wedge\forall\boldsymbol{z}\,(A(\boldsymbol{z})\to B(\boldsymbol{z}))$.

Similarly for the sequents:

Definition 4.2 Take $\boldsymbol{z}=(z_1,\dots,z_m)$ be the sequence of all free variables of the the sequent $A\Rightarrow B$ in the appearing order. Then

$$x\,\mathbf{q}^{\mathrm{PR}}(A\Rightarrow B) \equiv \mathrm{PR}(x)\wedge\forall y\,\boldsymbol{z}\,\big(y\,\mathbf{q}^{\mathrm{PR}}A(\boldsymbol{z})\to\varphi_x(y,\boldsymbol{z})\,\mathbf{q}^{\mathrm{PR}}B(\boldsymbol{z})\big)\wedge(A\to B).$$

The obvious property of **q**-realizability is:

Lemma 4.3 *For any formula A, $\mathbb{N}\vDash(n\,\mathbf{q}^{\mathrm{PR}}A)\to A$.*

The proof of soundness of BA to $\mathbf{r}^{\mathrm{PR}}$ works "as usual" ([29]) also for $\mathbf{q}^{\mathrm{PR}}$-realizability:

Theorem 4.4 *For all sequents $A\Rightarrow B$, if BA $\vdash A\Rightarrow B$, then for some natural n, $\mathbb{N}\vDash n\,\mathbf{q}^{\mathrm{PR}}\big(A\Rightarrow B\big)$.*

Finally, we obtain the following immediate application:

Corollary 4.5 *For every formula $A(\boldsymbol{x},y)$ with the presented free variables there is a (unary) primitive recursive function f such that*

(i) *if* BA $\vdash\ \Rightarrow\forall\boldsymbol{x}(\top\to\exists y\,A(\boldsymbol{x},y))$*, then* $\mathbb{N}\vDash\forall\boldsymbol{x}\,A(\boldsymbol{x},f(\boldsymbol{x}))$*,*

(ii) *if* BA $\vdash\ \Rightarrow\exists y\,A(\boldsymbol{x},y)$*, then* $\mathbb{N}\vDash\forall\boldsymbol{x}\,A(\boldsymbol{x},f(\boldsymbol{x}))$*,*

where $f(\boldsymbol{x})$ for a sequence $\boldsymbol{x}=(x_1,x_2,\dots,x_m)$ means $f(\langle x_1,\langle x_2,\dots\langle x_{m-1},x_m\rangle\rangle\dots\rangle)$.

Proof. (i) If BA $\vdash\ \Rightarrow\forall\boldsymbol{x}\,(\top\to\exists y\,A(\boldsymbol{x},y))$, then, by Theorem 3.5, there is an $n\in\mathbb{N}$ such that

$$\mathbb{N}\vDash n\,\mathbf{q}^{\mathrm{PR}}(\Rightarrow\forall\boldsymbol{x}\,(\top\to\exists y\,A(\boldsymbol{x},y))).$$

So, $\mathbb{N}\vDash\mathrm{PR}(n)\wedge\varphi_n(0)\,\mathbf{q}^{\mathrm{PR}}\forall\boldsymbol{x}(\top\to\exists y\,A(\boldsymbol{x},y))$, since $0\,\mathbf{q}^{\mathrm{PR}}\top$, hence

$$\mathbb{N}\vDash\mathrm{PR}(\varphi_n(0)) \text{ and } \mathbb{N}\vDash\forall\boldsymbol{x}\,(\varphi_{\varphi_n(0)}(0,\boldsymbol{x})\,\mathbf{q}^{\mathrm{PR}}\exists y\,A(\boldsymbol{x},y)).$$

Put $m=\varphi_n(0)$. Then $\mathbb{N}\vDash\forall\boldsymbol{x}\,(\pi_2\varphi_m(0,\boldsymbol{x})\,\mathbf{q}^{\mathrm{PR}}A(\boldsymbol{x},\pi_1\varphi_m(0,\boldsymbol{x})))$. Thus, by Lemma 4.3, $\mathbb{N}\vDash\forall\boldsymbol{x}\,A(\boldsymbol{x},f(\boldsymbol{x}))$, where $f(\boldsymbol{x})=\pi_1\varphi_m(0,\boldsymbol{x})$ is primitive recursive.

(ii) Suppose BA $\vdash\ \Rightarrow\exists y\,A(\boldsymbol{x},y)$. Similar to the case (i), by Theorem 3.5 we have $\mathbb{N}\vDash n\,\mathbf{q}^{\mathrm{PR}}(\Rightarrow\exists y\,A(\boldsymbol{x},y))$ for an n such that φ_n is primitive recursive. Hence $\mathbb{N}\vDash\forall\boldsymbol{x}\,(\varphi_n(0,\boldsymbol{x})\,\mathbf{q}^{\mathrm{PR}}\exists y\,A(\boldsymbol{x},y))$. Then by the definition of $\mathbf{q}^{\mathrm{PR}}$, we get $\mathbb{N}\vDash\forall\boldsymbol{x}\,(\pi_2\varphi_n(0,\boldsymbol{x})\,\mathbf{q}^{\mathrm{PR}}A(\boldsymbol{x},\pi_1\varphi_n(0,\boldsymbol{x})))$, and so by Lemma 4.3, $\mathbb{N}\vDash\forall\boldsymbol{x}\,A(\boldsymbol{x},f(\boldsymbol{x}))$, where $f(\boldsymbol{x})=\pi_1\varphi_n(0,\boldsymbol{x})$ is primitive recursive. □

Remark 4.6 There are a Π_2-sentence $\forall \boldsymbol{x}\,(\top \rightarrow \exists y\, \Gamma(\boldsymbol{x}, y))$ and a Σ_1-formula $\exists y\, \Gamma(\boldsymbol{x}, y)$ with free variables $\boldsymbol{x}$, which are provable in PA (and in HA) but not in BA: Let Γ be a defining Σ_1-formula of the Ackermann function such that HA proves its totality (see e. g. [29]), then since the Ackermann function is not primitive recursive, neither $\Rightarrow \forall \boldsymbol{x}\,(\top \rightarrow \exists y\, \Gamma(\boldsymbol{x}, y))$ nor $\Rightarrow \exists y\, \Gamma(\boldsymbol{x}, y)$ can be proven in BA.

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