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CLASSROOM NOTE

# Proving concurrency by loci

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**ABSTRACT**
A fascinating and catchy method for proving that a number of special lines concur is using the concept of locus. This is now the classical method for proving the concurrency of the internal angle bisectors and perpendicular side bisectors of a triangle. In this paper, we prove the concurrency of the altitudes and the medians by showing that they are loci of some interesting points. Our proofs for these ancient theorems seem to be new. We also provide loci method proofs for the concurrency theorems of Ceva and Carnot.

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## 1. Introduction

A marvellous observation in the geometry of triangles is the fact that there are some special lines in triangles that are *concurrent* (i.e. the lines meet at one point). The proofs of some of these theorems (specially that of the concurrency of altitudes and medians) could be forgotten if one has been away from geometry for long. Yet, a proof via *loci* (this is the Latin plural of *locus*, the location of all the points that share a certain property) may be easier to remember, even after many years. An example, maybe the simplest one, is recalling that a perpendicular side bisector of $BC$ in $\triangle ABC$ is the locus of all the points $X$ such that $|XB| = |XC|$. This means that a point $X$ lies on the perpendicular side bisector of $BC$ if and only if $X$ has equal distances from the points $B$ and $C$. Similarly, an internal angle bisector of $\angle A$ is the locus of all the points $X$ inside $\triangle ABC$ such that $X$ has equal distances from the sides $AB$ and $AC$. Thus, the three internal angle bisectors as well as the three perpendicular side bisectors of every triangle are concurrent. These theorems are proved in Euclid's *Elements* (as Propositions 4 and 5 of the book IV); see Hajja and Martini (2013, §2) or Ostermann and Wanner (2012, §§ 4.3). The concurrency of the altitudes and the medians do not appear in the *Elements* of Euclid, though they are classical theorems by now. Some believe that Archimedes knew the concurrency of the medians, see Ostermann and Wanner (2012, p. 84), and two proofs for the concurrency of the altitudes are attributed to Newton (1642–1726) and Gauss (1777–1855); see Hajja and Martini (2013, Proofs #2 & #1). In this paper, we prove these theorems, and also the concurrency theorems of G. Ceva (1647–1734) and L. Carnot (1753–1823), by using loci.

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***Dedicated to the loving memory of Behrouz Meshginghalam (1944–2013), a devoted Maths teacher.***

## 2. Altitudes and perpendiculars

Gauss's proof for the concurrency of the altitudes makes an indirect use of some loci. It actually shows that the altitudes of a triangle are perpendicular side bisectors of another triangle, and so are concurrent (being the loci of some points). Two other proofs of this theorem, Hajja and Martini (2013, Proofs #4 & #5), show that the altitudes of a triangle are internal angle bisectors of some other triangles. Here, we prove the concurrency of the altitudes by directly showing them to be some loci. The following theorem essentially appears in Petersen (1879, p. 10: **h.**). In its proof, we have considered the case where the altitude lies inside the triangle, cf. Hajja and Martini (2013, Lemma); other cases can be dealt with similarly.

**Theorem 2.1 (Each Altitude is a Locus):** *The locus of all the points X on the plane such that* $|XB|^2 - |XC|^2 = |AB|^2 - |AC|^2$ *is (the extended line of) the altitude AH (see Figure 1).*

***Proof:*** If $X$ is on the altitude $AH$, then apply Pythagoras' theorem to the four right triangles $\triangle ABH$, $\triangle AHC$, $\triangle XBH$, and $\triangle XHC$ as follows:

$$\begin{aligned} |XB|^2 - |XC|^2 &= (|BH|^2 + |HX|^2) - (|XH|^2 + |HC|^2) \\ &= |BH|^2 - |HC|^2 \\ &= (|BH|^2 + |HA|^2) - (|AH|^2 + |HC|^2) \\ &= |AB|^2 - |AC|^2. \end{aligned}$$

Now, suppose that we have $|XB|^2 - |XC|^2 = |AB|^2 - |AC|^2$; draw a perpendicular line from $X$ to $BC$ and assume that it meets $BC$ at $Y$. By Pythagoras' theorem,

$$\begin{aligned} |XB|^2 - |XC|^2 &= (|BY|^2 + |YX|^2) - (|XY|^2 + |YC|^2) \\ &= |BY|^2 - |YC|^2 \\ &= (|BY| + |YC|) \cdot (|BY| - |YC|) \\ &= |BC| \cdot (|BY| - |YC|). \end{aligned}$$

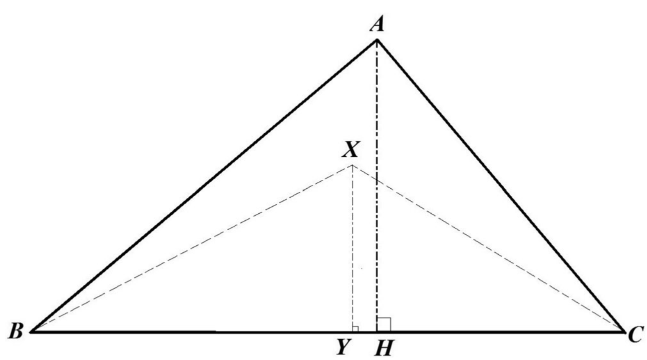


**Figure 1.** $X$ lies on $AH \iff Y = H$.

Since for a similar reason we have $|AB|^2 - |AC|^2 = |BC| \cdot (|BH| - |HC|)$, from the presumed assumption $|XB|^2 - |XC|^2 = |AB|^2 - |AC|^2$ and the above equality we get

$$\begin{cases} |BY| + |YC| = |BH| + |HC| \; (= |BC|), \\ |BY| - |YC| = |BH| - |HC| \; (= (|AB|^2 - |AC|^2)/|BC|). \end{cases}$$

Thus, $Y = H$, and so $X$ lies on $AH$. ■

**Corollary 2.2 (Altitudes Concur):** *The altitudes of a triangle are concurrent.*

***Proof:*** If $X$ is the intersection of the altitudes drawn from $B$ and $C$ in $\triangle ABC$, then by Theorem 2.1 we have

$$\begin{cases} |XA|^2 - |XC|^2 = |AB|^2 - |BC|^2, \\ |XB|^2 - |XA|^2 = |BC|^2 - |AC|^2. \end{cases}$$

Thus, $|XB|^2 - |XC|^2 = |AB|^2 - |AC|^2$, which results by adding the two sides of the above equations. So, by Theorem 2.1, $X$ lies on the altitude drawn from $A$ too. ■

The above proof does not appear among the 12 proofs for the concurrency of the altitudes listed by Hajja and Martini (2013). The concurrency of the altitudes (Corollary 2.2) as well as the concurrency of the perpendicular side bisectors are two special cases of Carnot's Concurrency Theorem, which can be proved by using loci as follows.

**Theorem 2.3 (Each Perpendicular is a Locus):** *Let $H$ be a point on and inside the line segment $BC$. The locus of all the points $X$ such that $|XB|^2 - |XC|^2 = |BH|^2 - |HC|^2$ is the line perpendicular to $BC$ at $H$ (see Figure 1).*

***Proof:*** If $X$ is on the line perpendicular to $BC$ with foot $H$, then we showed in the proof of Theorem 2.1 that the equality $|XB|^2 - |XC|^2 = |BH|^2 - |HC|^2$ holds. Conversely, if for a point $X$, $|XB|^2 - |XC|^2 = |BH|^2 - |HC|^2$ holds, then assume that the line perpendicular to $BC$ from $X$ meets $BC$ at $Y$. Therefore, similar to the proof of Theorem 2.1, we can show that $|BY| + |YC| = |BH| + |HC|$ and $|BY| - |YC| = |BH| - |HC|$. Thus, $Y = H$ and so the point $X$ lies on the perpendicular line to $BC$ with foot $H$. ■

**Corollary 2.4 (Carnot's Concurrency Theorem):** *Let $A', B', C'$ be some points on and inside, respectively, the sides $BC$, $AC$, $AB$ of $\triangle ABC$. The respective perpendiculars to the sides at the points $A', B', C'$ are concurrent if and only if Carnot's identity holds true for them: $|AB'|^2 + |BC'|^2 + |CA'|^2 = |AC'|^2 + |CB'|^2 + |BA'|^2$; or equivalently, the following equality holds: $(|AB'|^2 - |B'C|^2) + (|CA'|^2 - |A'B|^2) + (|BC'|^2 - |C'A|^2) = 0$ (see Figure 2).*

***Proof:*** Let $X$ be the intersection of the perpendiculars to $AB$ and $AC$ with, respectively, the feet $C'$ and $B'$. We have, by Theorem 2.3, $|XB|^2 - |XA|^2 = |BC'|^2 - |C'A|^2$ and $|XA|^2 - |XC|^2 = |AB'|^2 - |B'C|^2$. Thus, by adding the two sides of these equations, we get $|XB|^2 - |XC|^2 = (|BC'|^2 - |C'A|^2) + (|AB'|^2 - |B'C|^2)$. Now, by Theorem 2.3, $X$ lies on the perpendicular line with foot $A'$, if and only if $|XB|^2 - |XC|^2 = |BA'|^2 - |A'C|^2$, if and only if $(|BA'|^2 - |A'C|^2) = (|BC'|^2 - |C'A|^2) + (|AB'|^2 - |B'C|^2)$. ■

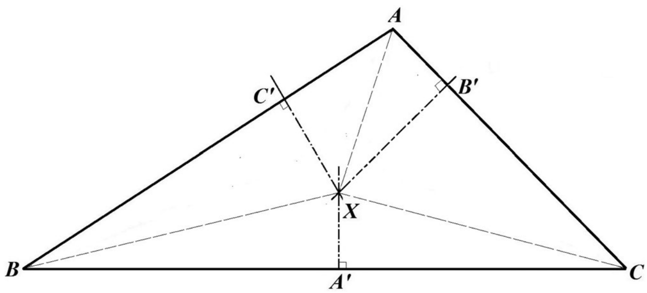


**Figure 2.** Carnot's Theorem.

Actually, Hajja and Martini (2013, Proof #7) infers the concurrence of the altitudes from Carnot's concurrency theorem. As a matter of fact, the altitudes are loci of some other, trigonometric, kind.

**Proposition 2.5 (Each Altitude is a(nother kind of) Locus):** *Let $H$ be a point on and inside the line segment $BC$. The locus of all the points $X$ on the plane such that $\frac{\text{cotg}(\angle XBC)}{\text{cotg}(\angle XCB)} = \frac{|BH|}{|HC|}$ is the line perpendicular to $BC$ at $H$ (see Figure 1).*

***Proof:*** If $X$ is on the line perpendicular to $BC$ with foot $H$, then $\text{cotg}(\angle XBC) = \frac{|BH|}{|XH|}$ and $\text{cotg}(\angle XCB) = \frac{|HC|}{|XH|}$; so, $\frac{\text{cotg}(\angle XBC)}{\text{cotg}(\angle XCB)} = \frac{|BH|}{|HC|}$. If, on the other hand, $X$ is a point on the plane for which $\frac{\text{cotg}(\angle XBC)}{\text{cotg}(\angle XCB)} = \frac{|BH|}{|HC|}$ holds, then draw a perpendicular line to $BC$ from $X$ to meet it at $Y$. Then, similar to what we saw above, $\frac{\text{cotg}(\angle XBC)}{\text{cotg}(\angle XCB)} = \frac{|BY|}{|YC|}$. Therefore, $\frac{|BH|}{|HC|} = \frac{|BY|}{|YC|}$, and so, similar to the proof of Theorem 2.1, we can show that $Y = H$. So, $X$ lies on the perpendicular line to $BC$ with foot $H$. ∎

Now, we can give another locus-method proof for the concurrency of the altitudes.

**An Alternative Proof for Corollary 2.2:** By Proposition 2.5, for the altitude $AH$ we have $\frac{\text{cotg}(\angle B)}{\text{cotg}(\angle C)} = \frac{|BH|}{|HC|}$. Let $X$ be the intersection of the altitudes drawn from $B$ and $C$. Then $\angle XBC = 90^\circ - \angle C$ and $\angle XCB = 90^\circ - \angle B$. Thus, $\frac{\text{cotg}(\angle XBC)}{\text{cotg}(\angle XCB)} = \frac{\text{cotg}(90^\circ - \angle C)}{\text{cotg}(90^\circ - \angle B)} = \frac{\text{tg}(\angle C)}{\text{tg}(\angle B)} = \frac{\text{cotg}(\angle B)}{\text{cotg}(\angle C)} = \frac{|BH|}{|HC|}$. Therefore, by Proposition 2.5, the point $X$ lies on the altitude $AH$ too.

## 3. Medians and Cevians

The following theorem essentially appears in Petersen (1879, pp. 9–10: **g.** & *App. 1*). Let $\mathcal{S}_{\mathtt{F}}$ denote the area (surface) of a figure F.

**Theorem 3.1 (Each Median is a Locus):** *The locus of all the points $X$ inside $\triangle ABC$ such that the triangles $\triangle AXB$ and $\triangle AXC$ have equal areas is the median $AM$ (see Figure 3, by taking $A' = M$).*

***Proof:*** If $X$ lies on the median $AM$, then since $M$ is the midpoint of $BC$, we have $\mathcal{S}_{\triangle ABM} = \mathcal{S}_{\triangle AMC}$, and also $\mathcal{S}_{\triangle XBM} = \mathcal{S}_{\triangle XMC}$. So, by subtracting the two sides of the

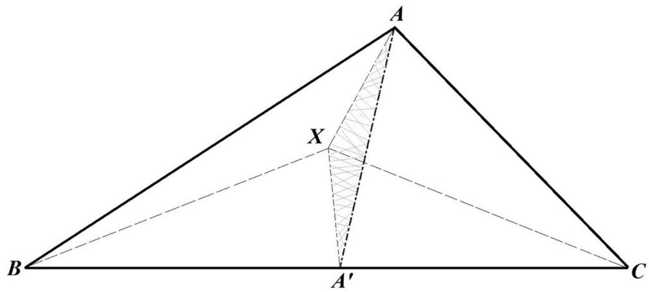


**Figure 3.** $X$ lies on $AA'$ $\iff$ $\mathcal{S}_{\triangle AXA'} = \mathbf{0}$.

equations, we get $\mathcal{S}_{\triangle AXB} = \mathcal{S}_{\triangle ABM} - \mathcal{S}_{\triangle XBM} = \mathcal{S}_{\triangle AMC} - \mathcal{S}_{\triangle XMC} = \mathcal{S}_{\triangle AXC}$. Now, for the converse implication, suppose that $\mathcal{S}_{\triangle AXB} = \mathcal{S}_{\triangle AXC}$. If $X$ does not lie on the line $AM$, then $X$ is either inside $\triangle ABM$ or inside $\triangle AMC$. Assume, without loss of generality, that $X$ is inside $\triangle ABM$ (see Figure 3). By the assumption $\mathcal{S}_{\triangle AXB} = \mathcal{S}_{\triangle AXC}$, we have $\mathcal{S}_{\triangle AXC} = \frac{1}{2}\mathcal{S}_{\lozenge ABXC}$. Since $M$ is the midpoint of $BC$, we have $\mathcal{S}_{\triangle AMC} = \frac{1}{2}\mathcal{S}_{\triangle ABC}$ and $\mathcal{S}_{\triangle XMC} = \frac{1}{2}\mathcal{S}_{\triangle XBC}$. Thus, $\mathcal{S}_{\lozenge AXMC} = \mathcal{S}_{\triangle AXC} + \mathcal{S}_{\triangle XMC} = \frac{1}{2}(\mathcal{S}_{\lozenge ABXC} + \mathcal{S}_{\triangle XBC}) = \frac{1}{2}\mathcal{S}_{\triangle ABC} = \mathcal{S}_{\triangle AMC}$. Therefore, $\mathcal{S}_{\triangle AXM} = \mathcal{S}_{\lozenge AXMC} - \mathcal{S}_{\triangle AMC} = 0$; so, $X$ lies on $AM$. ■

As a matter of fact, the locus of all the points $X$ *on the plane* with $\mathcal{S}_{\triangle AXB} = \mathcal{S}_{\triangle AXC}$ consists of two lines: one the extended line of the median $AM$, and the other the line drawn from $A$ parallel to $BC$ (see also the link https://t.ly/b1cT of `math.stackexchange.com`).

**Corollary 3.2 (Medians Concur):** *The medians of a triangle are concurrent.*

***Proof:*** If $X$ is the intersection of the medians drawn from $B$ and $C$ in $\triangle ABC$, then by Theorem 3.1 we have $\mathcal{S}_{\triangle BXA} = \mathcal{S}_{\triangle BXC}$ and $\mathcal{S}_{\triangle CXA} = \mathcal{S}_{\triangle CXB}$. Thus, $\mathcal{S}_{\triangle AXB} = \mathcal{S}_{\triangle AXC}$, and so by Theorem 3.1, the point $X$ lies on the median drawn from $A$ too. ■

It is now known that the concurrency of the internal angle bisectors, altitudes (Corollary 2.2), and medians (Corollary 3.2) are special cases of Ceva's Concurrency Theorem, which can also be proved by using loci. Let us recall that a Cevian is a line segment that connects a vertex of a triangle to a point on the opposite side.

**Theorem 3.3 (Each Cevian is a Locus):** *Let $A'$ be a point on and inside $BC$. The Cevian $AA'$ is the locus of all the points $X$ inside $\triangle ABC$ such that $\frac{\mathcal{S}_{\triangle AXB}}{\mathcal{S}_{\triangle AXC}} = \frac{|BA'|}{|A'C|}$ (see Figure 3).*

***Proof:*** Suppose, first, that $X$ lies on $AA'$. Then we have $\frac{\mathcal{S}_{\triangle ABA'}}{\mathcal{S}_{\triangle AA'C}} = \frac{|BA'|}{|A'C|}$, and $\frac{\mathcal{S}_{\triangle XBA'}}{\mathcal{S}_{\triangle XA'C}} = \frac{|BA'|}{|A'C|}$. By Proposition 19 in the Book V of Euclid's *Elements* ($f = \frac{a}{b} = \frac{c}{d} \Rightarrow f = \frac{a-c}{b-d}$, where $b \neq d$) we have $\frac{|BA'|}{|A'C|} = \frac{\mathcal{S}_{\triangle ABA'} - \mathcal{S}_{\triangle XBA'}}{\mathcal{S}_{\triangle AA'C} - \mathcal{S}_{\triangle XA'C}} = \frac{\mathcal{S}_{\triangle AXB}}{\mathcal{S}_{\triangle AXC}}$. Now, second, suppose that $\frac{\mathcal{S}_{\triangle AXB}}{\mathcal{S}_{\triangle AXC}} = \frac{|BA'|}{|A'C|}$ holds. If $X$ is not on $AA'$, then without loss of generality we can assume that $X$ is inside $\triangle ABA'$. By Proposition 18 in the Book V of Euclid's *Elements* ($\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a+b}{b} = \frac{c+d}{d}$) we have $\frac{\mathcal{S}_{\Diamond ABXC}}{\mathcal{S}_{\triangle AXC}} = \frac{|BC|}{|A'C|}$. On the other hand, we also have $\frac{\mathcal{S}_{\triangle XBC}}{\mathcal{S}_{\triangle XA'C}} = \frac{|BC|}{|A'C|} = \frac{\mathcal{S}_{\triangle ABC}}{\mathcal{S}_{\triangle AA'C}}$. Therefore, $\mathcal{S}_{\Diamond AXA'C} = \mathcal{S}_{\triangle AXC} + \mathcal{S}_{\triangle XA'C} = \frac{|A'C|}{|BC|}(\mathcal{S}_{\Diamond ABXC} + \mathcal{S}_{\triangle XBC}) = \frac{|A'C|}{|BC|}\mathcal{S}_{\triangle ABC} = \mathcal{S}_{\triangle AA'C}$. Thus, $\mathcal{S}_{\triangle AXA'} = \mathcal{S}_{\Diamond AXA'C} - \mathcal{S}_{\triangle AA'C} = 0$; so, $X$ lies on $AA'$. ■

**Corollary 3.4 (Ceva's Concurrency Theorem):** *Let $A'$, $B'$, and $C'$ be on (and inside), respectively, the sides $BC$, $AC$, and $AB$ of $\triangle ABC$. The Cevians $AA'$, $BB'$ and $CC'$ are concurrent if and only if Ceva's identity holds: $|AB'| \cdot |BC'| \cdot |CA'| = |AC'| \cdot |CB'| \cdot |BA'|$; or equivalently, $\frac{|AB'|}{|B'C|} \cdot \frac{|CA'|}{|A'B|} \cdot \frac{|BC'|}{|C'A|} = 1$.*

***Proof*:** If $X$ is the intersection of $BB'$ and $CC'$, then by Theorem 3.3 we have $\frac{\mathcal{S}_{\triangle BXA}}{\mathcal{S}_{\triangle BXC}} = \frac{|AB'|}{|B'C|}$ and $\frac{\mathcal{S}_{\triangle CXB}}{\mathcal{S}_{\triangle CXA}} = \frac{|BC'|}{|C'A|}$. Thus, by multiplying the two sides of these equations, we get $\frac{\mathcal{S}_{\triangle AXB}}{\mathcal{S}_{\triangle AXC}} = \frac{|AB'|}{|B'C|} \cdot \frac{|BC'|}{|C'A|}$. Now, by Theorem 3.3, $X$ lies on $AA'$, if and only if $\frac{\mathcal{S}_{\triangle AXB}}{\mathcal{S}_{\triangle AXC}} = \frac{|BA'|}{|A'C|}$, if and only if $\frac{|BA'|}{|A'C|} = \frac{|AB'|}{|B'C|} \cdot \frac{|BC'|}{|C'A|}$. ■

By a trigonometric version of Ceva's Concurrency Theorem, and using the Law of Sines, one can show that the medians are some other kind of loci too. Hence, we can give an alternative proof for Corollary 3.2 again by using loci.

**Proposition 3.5 (Each Median is a(nother kind of) Locus):** *The locus of all the points $X$ inside $\triangle ABC$ with $\frac{\sin(\angle XAB)}{\sin(\angle XAC)} = \frac{|AB|^{-1}}{|AC|^{-1}}$ is the median $AM$ (see Figure 3, by taking $A' = M$).*

***Proof*:** If $X$ lies on $AM$, then by the law of sines we have $\frac{\sin(\angle XAB)}{|BM|} = \frac{\sin(\angle AMB)}{|AB|}$ (in $\triangle AMB$) and $\frac{\sin(\angle XAC)}{|MC|} = \frac{\sin(\angle AMC)}{|AC|}$ (in $\triangle AMC$). So, from $|MB| = |MC|$ and $\sin(\angle AMB) = \sin(\angle AMC)$, we have $\frac{\sin(\angle XAB)}{\sin(\angle XAC)} = \frac{|AB|^{-1}}{|AC|^{-1}}$. If, conversely, we have $\frac{\sin(\angle XAB)}{\sin(\angle XAC)} = \frac{|AB|^{-1}}{|AC|^{-1}}$, then prolong $AX$ to meet $BC$ at $M'$. Then by the law of sines we have $\frac{\sin(\angle XAB)}{|BM'|} = \frac{\sin(\angle AM'B)}{|AB|}$ (in $\triangle AM'B$) and $\frac{\sin(\angle XAC)}{|M'C|} = \frac{\sin(\angle AM'C)}{|AC|}$ (in $\triangle AM'C$). So, $\frac{\sin(\angle XAB)}{\sin(\angle XAC)} = \frac{|BM'|}{|M'C|} \cdot \frac{|AB|^{-1}}{|AC|^{-1}}$. Thus, from the assumption $\frac{\sin(\angle XAB)}{\sin(\angle XAC)} = \frac{|AB|^{-1}}{|AC|^{-1}}$, we get $|BM'| = |M'C|$. Therefore, $M' = M$; so, $X$ lies on $AM$. ■

Finally, we can give another locus-method proof for the concurrency of the medians.

**An Alternative Proof for Corollary 3.2:** Let the medians through $B$ and $C$ meet each other at $X$. By Proposition 3.5, we have (1) $\frac{\sin(\angle XBA)}{\sin(\angle XBC)} = \frac{|AB|^{-1}}{|BC|^{-1}}$ and (2) $\frac{\sin(\angle XCB)}{\sin(\angle XCA)} = \frac{|BC|^{-1}}{|AC|^{-1}}$. Also, by the law of sines we have (3) $\frac{\sin(\angle XAB)}{|BX|} = \frac{\sin(\angle XBA)}{|AX|}$ (in $\triangle XAB$), (4) $\frac{\sin(\angle XAC)}{|XC|} = \frac{\sin(\angle XCA)}{|AX|}$ (in $\triangle XAC$), and (5) $\frac{\sin(\angle XBC)}{|XC|} = \frac{\sin(\angle XCB)}{|XB|}$ (in $\triangle XBC$). Therefore,

$$\frac{\sin(\angle XAB)}{\sin(\angle XAC)} = \frac{|BX| \cdot \sin(\angle XBA)}{|XC| \cdot \sin(\angle XCA)} \quad \text{by (3) and (4)}$$

$$= \frac{\sin(\angle XCB)}{\sin(\angle XBC)} \cdot \frac{\sin(\angle XBA)}{\sin(\angle XCA)} \quad \text{by (5)}$$

$$= \frac{\sin(\angle XBA)}{\sin(\angle XBC)} \cdot \frac{\sin(\angle XCB)}{\sin(\angle XCA)}$$

$$= \frac{|AB|^{-1}}{|AC|^{-1}} \quad \text{by (1) and (2).}$$

So, by Proposition 3.5, the point $X$ lies on the median through $A$ too.

## 4. Conclusions

The internal angle bisectors of a triangle concur because the internal bisector of an angle is the locus of all the points inside the triangle that are equidistant from the sides of the angle. The essentially same argument shows that an internal bisector of an angle concurs with the external bisectors of the two other angles. The perpendicular side bisectors of a triangle concur because each of them is the locus of all the points that are equidistant from the two vertices of the side. These two theorems appear in Euclid's *Elements*, and are proved in that book by using loci. Two other now-classical concurrency theorems, that of the altitudes and the medians, do not appear there; though, the ancient Greeks had all the tools for proving those theorems. Could a reason for this exclusion be that no proof by using loci was known for them? In this paper, we proved these two theorems by the loci method. We noted that the altitude $AH$ of $\triangle ABC$ is the locus of all the points $X$ on the plane such that $|XB|^2 - |XC|^2$ is the fixed value $|AB|^2 - |AC|^2$ (Theorem 2.1); and the median $AM$ is the locus of all the points $X$ inside $\triangle ABC$ such that the triangles $\triangle AXB$ and $\triangle AXC$ have equal areas (Theorem 3.1). Thus, we presented proofs for the concurrence of the altitudes (Corollary 2.2) and the medians (Corollary 3.2) by using loci. As a generalisation, we showed (Theorem 2.3) that a perpendicular line to $BC$ from a point $A'$ on it is the locus of all the points $X$ such that $|XB|^2 - |XC|^2$ is the fixed value $|BA'|^2 - |A'C|^2$. As a result, Carnot's concurrency theorem follows (Corollary 2.4): for the points $A', B', C'$ on (and inside), respectively, the sides $BC$, $AC$, $AB$ of $\triangle ABC$, the perpendicular lines from those points to the corresponding sides are concurrent if and only if Carnot's identity $|AB'|^2 + |BC'|^2 + |CA'|^2 = |AC'|^2 + |CB'|^2 + |BA'|^2$ holds. Also, we showed (Theorem 3.3) that each Cevian $AA'$ is the locus of all the points $X$ inside $\triangle ABC$ such that the ratio of the area of $\triangle AXB$ to the area of $\triangle AXC$ is the fixed fraction $|BA'|/|A'C|$. So, Ceva's concurrency theorem follows as a result (Corollary 3.4): the Cevians $AA'$, $BB'$, $CC'$ in $\triangle ABC$ concur if and only if Ceva's identity $|AB'| \cdot |BC'| \cdot |CA'| = |AC'| \cdot |CB'| \cdot |BA'|$ holds. The altitudes (Proposition 2.5) and the medians (Proposition 3.5) were proved to be some other (trigonometric) kind of loci, and so we presented alternative proofs for their concurrence by using loci.

## Disclosure statement

No potential conflict of interest was reported by the author.

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## Exercises

I apologize to the senior readers for putting some exercises at the end.

(1) Show that if $AH$ is an altitude and $AD$ is an internal angle bisector in $\triangle ABC$, then $\frac{|BH|}{|HC|} = \frac{\mathrm{cotg}(\angle B)}{\mathrm{cotg}(\angle C)}$ and $\frac{|BD|}{|DC|} = \frac{\mathrm{cosec}(\angle B)}{\mathrm{cosec}(\angle C)}$. Deduce Ceva's identity for (the feet of) the altitudes and the internal angle bisectors. Deduce Carnot's identity for (the feet of) the altitudes by noting that, e.g. $|BH| = |BC|\frac{\mathrm{cotg}(\angle B)}{\mathrm{cotg}(\angle B)+\mathrm{cotg}(\angle C)}$.

(2) Let $A'$ be on the side $BC$ in $\triangle ABC$. Show that
 (I) $\frac{\mathcal{S}_{\triangle ABA'}}{\mathcal{S}_{\triangle AA'C}} = 1$ iff $A'$ is the foot of the median;
 (II) $\frac{\mathcal{S}_{\triangle ABA'}}{\mathcal{S}_{\triangle AA'C}} = \frac{|AB|}{|AC|}$ iff $A'$ is the foot of the internal angle bisector;
 (III) $\frac{\mathcal{S}_{\triangle ABA'}}{\mathcal{S}_{\triangle AA'C}} = \frac{|AB|}{|AC|} \cdot \frac{\cos(\angle B)}{\cos(\angle C)}$ iff $A'$ is the foot of the altitude.

(3) In Carnot's Concurrency Theorem, what happens if $B' = C$ and $C' = A$? What is then the relation of $A'$ to $H$, the foot of the altitude $AH$?

(4) Given two distinct points $A$ and $B$, find the locus of all the points $X$ such that the absolute value $\big|\,|XA|^2 - |XB|^2\big|$ is a fixed positive number $\mathfrak{c}$. What happens if $\mathfrak{c} \to 0$? What if $\mathfrak{c} = |AB|^2$? What happens if $\mathfrak{c} \to \infty$?

(5) Given $\triangle ABC$, find the locus of all the points $X$ on the plane such that the following fractions, each separately, is a fixed positive number $\mathfrak{c}$. What happens if $\mathfrak{c} \to 0$? What if $\mathfrak{c} = 1$? What happens if $\mathfrak{c} \to \infty$?

$$\text{(i)}\ \frac{\mathcal{S}_{\triangle XAB}}{\mathcal{S}_{\triangle XAC}} \quad \text{(ii)}\ \frac{\sin(\angle XAB)}{\sin(\angle XAC)} \quad \text{(iii)}\ \frac{\cos(\angle XAB)}{\cos(\angle XAC)} \quad \text{(iv)}\ \frac{\mathrm{tg}(\angle XAB)}{\mathrm{tg}(\angle XAC)}$$

(6) Given two distinct points $A$ and $B$, find the locus of all the points $X$ such that the fraction $\frac{|XA|}{|XB|}$ is a fixed positive number $\mathfrak{c}$. What happens if $\mathfrak{c} \to 0$? What if $\mathfrak{c} = 1$? What happens if $\mathfrak{c} \to \infty$?

**Notation** Let $\mathsf{d}(X, \ell)$ denote the distance of the point $X$ from the line $\ell$.

(7) Show that, given two intersecting lines $\ell$ and $\ell'$, the locus of all points $X$ such that the fraction $\frac{\mathsf{d}(X,\ell)}{\mathsf{d}(X,\ell')}$ is a fixed number $\mathfrak{c}$ consists of two intersecting lines. What happens if $\mathfrak{c} \to 0$? What if $\mathfrak{c} = 1$? What happens if $\mathfrak{c} \to \infty$?Answer all the questions in the case that $\ell$ and $\ell'$ are parallel.

(8) Show that the median $AM$ is the locus of all the points $X$ inside $\triangle ABC$ such that $\frac{\mathsf{d}(X,AB)}{\mathsf{d}(X,AC)} = \frac{|AB|^{-1}}{|AC|^{-1}}$. Show that a Cevian $AA'$ is the locus of all the points $X$ inside $\triangle ABC$ such that $\frac{\mathsf{d}(X,AB)}{\mathsf{d}(X,AC)} = \frac{|AB|^{-1}|BA'|}{|AC|^{-1}|A'C|}$.

(9) In $\triangle ABC$, choose the points $N_a$, $N_b$, and $N_c$ on, respectively, the sides $BC$, $AC$, and $AB$, in a way that we have $|AN_b| = |AC|\frac{\mathrm{cotg}(\angle A/2)}{\mathrm{cotg}(\angle A/2)+\mathrm{cotg}(\angle C/2)}$, $|BN_c| = |AB|\frac{\mathrm{cotg}(\angle B/2)}{\mathrm{cotg}(\angle A/2)+\mathrm{cotg}(\angle B/2)}$, and finally $|CN_a| = |BC|\frac{\mathrm{cotg}(\angle C/2)}{\mathrm{cotg}(\angle B/2)+\mathrm{cotg}(\angle C/2)}$. By proving Ceva's identity, show that $AN_a$, $BN_b$, and $CN_c$

are concurrent. Also, prove Carnot's identity for $N_a, N_b, N_c$, and show that the perpendicular lines to the sides with feet $N_a$, $N_b$, and $N_c$ are concurrent at the incenter of $\triangle ABC$. Prove that $N_a, N_b, N_c$ are the tangency points of the incircle of the triangle $\triangle ABC$.

(10) Let us call the points $(A', B', C')$ on, respectively, the sides $(BC, AC, AB)$ of $\triangle ABC$, a Ceva-Carnot triple, when both Ceva's identity and Carnot's identity hold (thus, the Cevians $AA'$, $BB'$, and $CC'$ are concurrent, and so are the perpendicular lines to the sides with feet $A', B', C'$). Examples of such triples include the midpoints of the sides, and the tangency points of the incircle. Let $\mathbf{A}'$ be fixed on $BC$. Show that either

(1) there are no points $(B', C')$ such that $(\mathbf{A}', B', C')$ is a Ceva-Carnot triple, or
(2) there is exactly one couple of points $(B', C')$ such that $(\mathbf{A}', B', C')$ is a Ceva-Carnot triple, or
(3) there are exactly two couples of points $(B', C')$ such that $(\mathbf{A}', B', C')$ are Ceva-Carnot triples, or
(4) there are infinitely many couples of points $(B', C')$ such that $(\mathbf{A}', B', C')$ are Ceva-Carnot triples.

Provide examples for each of the four cases.