Gödel–Rosser's Incompleteness Theorem, generalized and optimized for definable theories

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Abstract

Gödel–Rosser's Incompleteness Theorem is generalized by showing Π_{n+1}-incompleteness of any Σ_{n+1}-definable extension of Peano Arithmetic which is either Σ_n-sound or n-consistent. The optimality of this result is proved by presenting a complete, Σ_{n+1}-definable, Σ_{n-1}-sound, and $(n-1)$-consistent theory for any $n>0$. Though the proof of the incompleteness theorem for Σ_{n+1}-definable theories using the Σ_n-soundness assumption is constructive, it is shown that there is no constructive proof for the Incompleteness Theorem for Σ_{n+1}-definable theories using the n-consistency assumption, when $n>2$.

Keywords: Gödel's incompleteness theorem, recursive enumerability, definability, Rosser's Trick, Craig's Trick, Σ_n-soundness, n-consistency.

1 Introduction

By Gödel–Rosser's Incompleteness Theorem no recursively enumerable (RE) and consistent extension of Robinson's Arithmetic **Q** can be complete. A natural question is then if the incompleteness phenomenon holds for non-RE theories. Indeed, there exists a complete and Σ_2-definable extension of **Q** (see e.g. [9, p. 860–861]). So, the notion of consistency should also be generalized when one wishes to consider more general theories (which are not necessarily RE). Considering more general theories should be restricted, though, to definable ones, since e.g. the non-definable (by Tarski's Theorem) true arithmetic Th(ℕ) is complete. We consider two hierarchies of consistency notions: one is Σ_n*-soundness* and the other is n*-consistency*.
In this article, along with comparing the above mentioned two hierarchical notions (Proposition 3.2), we show that no Σ_{n+1}-definable extension of **Q** can be complete if it is Σ_n-sound (Theorem 2.5). For $n=0$, this is exactly Gödel–Rosser's Theorem (Σ_1-definable sets are exactly the RE sets and Σ_0-soundness is equivalent to simple consistency for extensions of **Q**). Moreover, its proof is constructive: for a given Σ_{n+1}-formula which defines a Σ_n-sound extension of **Q** one can construct a sentence independent from that theory. We will show the optimality of this result by presenting a Σ_{n-1}-sound and Σ_{n+1}-definable complete extension of **Q**, for any $n\geqslant 1$ (Theorem 2.6). For the notion of n-consistency, we will show the incompleteness of any Σ_{n+1}-definable and n-consistent extension of **Q**, non-constructively (Theorem 4.3). This result is also optimal, since it can be shown that there exists

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an $(n-1)$-consistent and Σ_{n+1}-definable complete extension of Q, for any $n \geqslant 1$. Its proof cannot be constructive, since we will show that, for $n > 2$, there exists no algorithm which for any given Σ_{n+1}-formula defining an n-consistent extension of Q outputs an independent sentence (Theorem 4.4). In conclusion, Gödel–Rosser's Theorem can be generalized to the incompleteness of Σ_{n+1}-definable and Σ_n-sound extensions of Q, *constructively*; and also to the incompleteness of Σ_{n+1}-definable and n-consistent extensions of Q, *non-constructively* (when $n > 2$).

1.1 Some background and history

Indeed, the ideas behind Theorems 2.3 and 2.5 'are not essentially new', as a referee suggested, 'but they had not been systematically presented before'. The Π_1-incompleteness of any consistent extension of Peano Arithmetic whose set of theorems is Δ_2-definable, was shown in [5, Theorem 2]. This was further generalized in [3, Theorem 2.8] by showing the Π_n-incompleteness of any consistent extension of Peano Arithmetic whose set of theorems is Δ_{n+1}-definable. It was also shown in [3, Theorem 2.5] that, for any $n \geqslant 2$, if the set of theorems of an n-consistent extension of Peano Arithmetic is Π_n-definable, then it is neither Π_{n-1}-complete nor complete (cf. Theorem 4.3 below). The Π_{n+1}-incompleteness of any consistent and Δ_{n+1}-definable extension of Π_n-Th($\mathbb{N}$), for any $n \geqslant 1$, is mentioned in [6, Exercise 9.12(d)]; it also appears in some other forms in [1, Theorems 2.1, 2.2] and [2, Proposition 4.1(2)]. In Theorem below 2.5, the condition of 'extending Π_n-Th($\mathbb{N}$)' is replaced with the weaker condition of 'being consistent with Π_n-Th($\mathbb{N}$)'.

2 Incompleteness of Σ_n-Sound Σ_{n+1}-Definable Theories

DEFINITION 2.1 (Definablility/soundness/completeness)
Let Γ be a class of formulas.
A theory T is called *definable* if a formula $\mathsf{Ax}_T(x)$ defines (the set of Gödel codes of) its axioms (in $\mathbb{N}$).
A theory is called Γ-*definable* when $\mathsf{Ax}_T(x) \in \Gamma$.
A theory T is called Γ-*sound* when for any $\varphi \in \Gamma$, if $T \vdash \varphi$ then $\mathbb{N} \models \varphi$.
A theory T is called Γ-*complete* when for any $\varphi \in \Gamma$, if $\mathbb{N} \models \varphi$ then $T \vdash \varphi$.

DEFINITION 2.2 (The arithmetical hierarchy)
The classes of formulas $\{\Pi_n\}_{n\in\mathbb{N}}$ and $\{\Sigma_n\}_{n\in\mathbb{N}}$ are defined in the standard way (see e.g. [6]). Let $\overline{\Pi}_n$ be the closure of Π_n under disjunction, conjunction, universal quantifiers and bounded existential quantifiers. Similarly, $\overline{\Sigma}_n$ is the closure of Σ_n under disjunction, conjunction, existential quantifiers and bounded universal quantifiers.

Let us note that every $(\overline{\Pi}_n$-$)\overline{\Sigma}_n$-formula is equivalent to a $(\Pi_n$-$)\Sigma_n$-formula in Peano Arithmetic (and in $\mathbb{N}$). Throughout the article, Π_n-Th($\mathbb{N}$), respectively, Σ_n-Th($\mathbb{N}$), denotes the set of all true Π_n-sentences, respectively, Σ_n-sentences, which (for $n \geqslant 1$) is definable by the Π_n-formula Π_n-$\mathsf{True}(x)$, respectively, the Σ_n-formula Σ_n-$\mathsf{True}(x)$ (see e.g. [6]).

THEOREM 2.3 ($\overline{\Pi}_{n+1}$-incompleteness of Π_n-definable and Σ_n-sound theories)
For any $n \geqslant 1$, if T is Σ_n-sound and $\mathsf{Ax}_T \in \Pi_n$, then there exists a (true) $\overline{\Pi}_{n+1}$-sentence $\boldsymbol{\gamma}$ independent from T.

PROOF. Since T is Σ_n-sound then the theory $T^*=T\cup\Pi_n\text{-Th}(\mathbb{N})$ is consistent. Suppose $\mathsf{Proof}(z,x)$ denotes a (Σ_0) proof predicate in (pure) classical logic, stating that z is a (Gödel code of a) proof for the sentence (with Gödel code) x, and consider the following proof predicate $\mathsf{Proof}_{T^*}(z,x)$ for T^*:

$$\exists w,u,t\leqslant z\big[z=\langle w,u,t\rangle\wedge\Pi_n\text{-}\mathsf{True}(w)\wedge\mathsf{ConjAx}_T(u)\wedge\mathsf{Proof}(t,w\wedge u\to x)\big]$$

where the formula $\mathsf{ConjAx}_T(x)$ states that 'x is the Gödel code of a formula which is a conjunction of some axioms of T', i.e. $x=\ulcorner\bigwedge_{i=1}^{\ell}\varphi_i\urcorner$ and $\bigwedge_{i=1}^{\ell}\mathsf{Ax}_T(\ulcorner\varphi_i\urcorner)$, and $\langle -,-,-\rangle$ is a suitable Σ_0-definable coding for triples. Let us note that $\mathsf{ConjAx}_T\in\Pi_n$ when $\mathsf{Ax}_T\in\Pi_n$ (and $n\geqslant 1$). By the diagonal lemma, noting that $T^*\supseteq\Sigma_1\text{-Th}(\mathbb{N})$, there exists a sentence $\boldsymbol{\gamma}$ such that

$$T^*\vdash\boldsymbol{\gamma}\leftrightarrow\forall z\big[\mathsf{Proof}_{T^*}(z,\ulcorner\boldsymbol{\gamma}\urcorner)\longrightarrow\exists z'<z\mathsf{Proof}_{T^*}(z',\ulcorner\neg\boldsymbol{\gamma}\urcorner)\big].$$

Now, by a classical Gödel–Rosser argument one can show that the sentence $\boldsymbol{\gamma}$ is ($\overline{\Pi}_{n+1}$ and true and) independent from T^* (and so from T). ■

Let us note that the question as to whether the independent sentence $\boldsymbol{\gamma}$ can be taken to be Π_{n+1} is left open here; however, when T contains Peano Arithmetic, $\boldsymbol{\gamma}$ is (provably) Π_{n+1}. For genuinely generalizing Gödel–Rosser's Theorem, cf. Tarski's Theorem on *the incompleteness of any definable and sound theory* in [7, Theorem 1, p. 97] (see also [10]), we need the following:

LEMMA 2.4 (Craig's Trick, generalized)
For any $n\geqslant 1$, any Σ_{n+1}-definable (arithmetical) theory is equivalent with a Π_n-definable theory.

PROOF. If $\mathsf{Ax}_T(x)=\exists x_1\cdots\exists x_m\theta(x,x_1,\cdots,x_m)$ with $\theta\in\Pi_n$ then $\mathsf{Ax}_T(x)\equiv\exists y\theta'(x,y)$ with $\theta'(x,y)=\exists x_1,\cdots,x_m\leqslant y\theta(x,x_1,\cdots,x_m)\in\overline{\Pi}_n$. Now, $T'=\{\psi\wedge(\overline{k}=\overline{k})\mid\mathbb{N}\models\theta'(\ulcorner\psi\urcorner,k)\}$ is equivalent with T and is $\overline{\Pi}_n$-definable by $\mathsf{Ax}_{T'}(x)\equiv\exists y,z\leqslant x\big(\theta'(y,z)\wedge\big[x=(y\wedge\ulcorner\overline{z}=\overline{z}\urcorner)\big]\big)$. Let us recall that any $\overline{\Pi}_n$-definable theory is also Π_n-definable. ■

THEOREM 2.5 ($\overline{\Pi}_{n+1}$-incompleteness of Σ_{n+1}-definable and Σ_n-sound theories)
For any $n\geqslant 1$, if T is Σ_n-sound and $\mathsf{Ax}_T\in\Sigma_{n+1}$, then there exists a (true) $\overline{\Pi}_{n+1}$-sentence independent from T.

PROOF. By Lemma 2.4, every Σ_{n+1}-definable theory is equivalent to a Π_n-definable one, and by Theorem 2.3 for every such Σ_n-sound theory there exists an independent $\overline{\Pi}_{n+1}$-sentence. ■

Let us note that the above theorem has a constructive proof; i.e. there exists an algorithm, which given a Σ_{n+1}-sentence $\mathsf{Ax}_T(x)$ that defines the set of axioms of a Σ_n-sound theory T, outputs an $\overline{\Pi}_{n+1}$-sentence $\boldsymbol{\gamma}_T$ independent of T. Gödel's original proof for his first Incompleteness Theorem could work assuming the Σ_1-soundness of a Σ_1-definable extension of $\mathbf{Q}$ (whose incompleteness was to be proved). Rosser's trick weakened the assumption of Σ_1-soundness to simple consistency, or equivalently (for extensions of $\mathbf{Q}$) to Σ_0-soundness. Thus, a direct generalization of Gödel's incompleteness would be the $\overline{\Pi}_{n+1}$-incompleteness of any Σ_{n+1}-definable and Σ_{n+1}-sound theory. Theorem 2.5 above, which shows the $\overline{\Pi}_{n+1}$-incompleteness of any Σ_{n+1}-definable and Σ_n-sound theory, is more of a Rosserian style. Below we show its optimality:

THEOREM 2.6 (Optimality of generalized Gödel–Rosser Theorem)
For any $n\geqslant 1$ there exists a theory which is Σ_{n+1}-definable, Σ_{n-1}-sound and complete (and contains $\mathbf{Q}$).

PROOF. That there exists a complete Σ_2-definable extension of $\mathbf{Q}$ is already known; see e.g. [9]. Here, we generalize this result to $\mathbf{Q}\cup\Pi_{n-1}\text{-Th}(\mathbb{N})$. Let S be $\mathbf{Q}$ when $n=1$ and be $\mathbf{Q}\cup\Pi_{n-1}\text{-Th}(\mathbb{N})$

when $n>1$ (note that Π_0-Th($\mathbb{N}$)$\subseteq$**Q**). The theory S can be completed by Lindenbaum's Lemma as follows: for an enumeration of all the sentences $\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \cdots$ take $T_0=S$, and let $T_{i+1}=T_i\cup\{\boldsymbol{\psi}_i\}$ if $T_i\cup\{\boldsymbol{\psi}_i\}$ is consistent, and let $T_{i+1}=T_i\cup\{\neg\boldsymbol{\psi}_i\}$ otherwise (if $T_i\cup\{\boldsymbol{\psi}_i\}$ is inconsistent). Then the theory $\widehat{T}=\bigcup_{j\in\mathbb{N}}T_j$ is a complete extension of S; below we show the Σ_{n+1}-definability of $\widehat{T}$. An enumeration of all the sentences can be defined by a Σ_0-formula stating 'x is the (Gödel number of the) u-th sentence': $\mathsf{Sent\text{-}List}(x,u)=\big[\mathsf{Sent}(u)\wedge x=u\big]\vee\big[\neg\mathsf{Sent}(u)\wedge x=\ulcorner 0=0\urcorner\big]$, where $\mathsf{Sent}(u)$ is a Σ_0-formula stating that u is (the Gödel code of) a sentence. Now, if the Σ_0-formula $\mathsf{Seq}(y)$ states that y is the Gödel code of a sequence and its length is denoted by $\ell en(y)$ and for any number $l<\ell en(y)$ the l-th member of y is denoted by $[y]_l$, then $\mathsf{Ax}_{\widehat{T}}(x)$ can be defined by

$$\exists y\Big[\mathsf{Seq}(y)\wedge[y]_{\ell en(y)-1}=x\wedge\big(\forall u<\ell en(y)\big[\mathsf{Sent}([y]_u)\big]\big)\wedge$$
$$\forall u<\ell en(y)\forall z\leqslant y\Big(\big(\mathsf{Sent\text{-}List}(z,u)\wedge\mathsf{Con}(S+\langle y\lfloor u\rangle+z)\longrightarrow[y]_u=z\big)$$
$$\wedge\big(\mathsf{Sent\text{-}List}(z,u)\wedge\neg\mathsf{Con}(S+\langle y\lfloor u\rangle+z)\longrightarrow[y]_u=\neg z\big)\Big)\Big],$$

where $\langle y\lfloor u\rangle$ denotes the initial segment of y with length u, and the formula $\mathsf{Con}(S+\langle y\lfloor u\rangle+z)$ is defined below, where q is the Gödel code of the conjunction of all the (finitely many) axioms of **Q** and the Σ_0-formula $\mathsf{ConjSeq}(v,y)$ states that v is (the Gödel code of) the conjunction of all the members of the sequence (with Gödel code) y:
for $n=1$, $\mathsf{Con}(S+\langle y\lfloor u\rangle+z)$ is $\forall v,w\big[\mathsf{ConjSeq}(v,\langle y\lfloor u\rangle)\rightarrow\neg\mathsf{Proof}(w,\neg[\mathrm{q}\wedge v\wedge z])\big]$, and
for $n>1$, $\mathsf{Con}(S+\langle y\lfloor u\rangle+z)$ is

$$\forall t,v,w\big[\mathsf{ConjSeq}(v,\langle y\lfloor u\rangle)\wedge\Pi_{n-1}\text{-}\mathsf{True}(t)\rightarrow\neg\mathsf{Proof}(w,\neg[\mathrm{q}\wedge v\wedge z\wedge t])\big].$$

Since $\mathsf{ConjSeq},\mathsf{Proof}\in\Pi_0$ and (for $n>1$) Π_{n-1}-$\mathsf{True}\in\Pi_{n-1}$ then $\mathsf{Con}\in\Pi_n$ and so $\mathsf{Ax}_{\widehat{T}}\in\overline{\Sigma}_{n+1}$, which is equivalent to $\mathsf{Ax}_{\widehat{T}}\in\Sigma_{n+1}$ (in $\mathbb{N}$). ■

3 *n*-Consistency Versus Σ_m-Soundness

Definition 3.1 (*n*-Consistency)
A theory T is called *ω-consistent* when for no formula ψ both (I) $T\vdash\neg\psi(\overline{n})$ for all $n\in\mathbb{N}$, and (II) $T\vdash\exists x\psi(x)$ hold together. It is called *n-consistent* when for no formula $\psi\in\Sigma_n$ with $\psi=\exists x\theta(x)$ and $\theta\in\Pi_{n-1}$ one has (i) $T\vdash\neg\theta(\overline{n})$ for all $n\in\mathbb{N}$, and (ii) $T\vdash\psi$.

Proposition 3.2 (*n*-consistency versus Σ_n-soundness)
(1) If a theory is Σ_n-sound, then it is *n*-consistent.
(2) If a Σ_{n-1}-complete theory is *n*-consistent, then it is Σ_n-sound.

Proof. (1) Assume $T\vdash\exists x\psi(x)$ for some Σ_n-sound theory T and some Π_{n-1}-formula ψ. By Σ_n-soundness, $\mathbb{N}\models\exists x\psi(x)$, so $\mathbb{N}\models\psi(m)$ for some $m\in\mathbb{N}$. Now, $\psi(\overline{m})\in\Pi_{n-1}$-Th($\mathbb{N}$), thus $T\nvdash\neg\psi(\overline{m})$ by the Σ_n-soundness of T (again).
(2) Assume $T\vdash\exists x\psi(x)$ for some Σ_{n-1}-complete and *n*-consistent theory T and $\psi\in\Pi_{n-1}$. By *n*-consistency, there exists some $m\in\mathbb{N}$ such that $T\nvdash\neg\psi(\overline{m})$. By Σ_{n-1}-completeness we have $\mathbb{N}\not\models\neg\psi(\overline{m})$; and so $\mathbb{N}\models\psi(\overline{m})$, whence $\mathbb{N}\models\exists x\psi(x)$. ■

Remark 3.3 (($n+1$)-consistency versus Σ_n-soundness)
For $n=0,1,2$ the notions of Σ_n-soundness and *n*-consistency are equivalent for Σ_1-complete theories (see [4, Theorems 5, 25, 30]). But for $n\geq3$ the *n*-consistency does not necessarily imply the

Σ_n-soundness. Even the notion of ω-consistency does not imply the Σ_3-soundness (see [4, Theorem 19] proved by Kreisel in 1955; cf. [7, Theorem 5, p. 101]). Also, for any n, there exists a Σ_n-sound theory which is not $(n+1)$-consistent: Pick any $\varphi \in \Pi_{n+1}$-Th($\mathbb{N}$) such that $U = \Pi_n$-Th($\mathbb{N}$)$\cup\{\neg\varphi\}$ is consistent (see e.g. [6, Exercise 9.12(d)]). It is clear that U is Σ_n-sound, and U is not $(n+1)$-consistent, for otherwise U would be Σ_{n+1}-sound by Proposition 3.2(2).

4 Incompleteness of n-Consistent Σ_{n+1}-Definable Theories

DEFINITION 4.1 (Deciding formulas)
We say that a theory T *decides* a formula ψ when either $T \vdash \psi$ or $T \vdash \neg\psi$. For a class Γ formulas, a theory T is called Γ*-deciding* when it can decide every sentence in Γ.

The following lemma generalizes [4, Theorem 20] which states that *the true arithmetic* Th($\mathbb{N}$) *is the only ω-consistent extension of Peano Arithmetic* (*indeed* **Q**) *that is complete*.

LEMMA 4.2 (Π_n-completeness of n-consistent and Π_n-deciding theories)
Any n-consistent and Π_n-deciding extension of **Q** is Π_n-complete.

PROOF. By induction on n. For $n=0$ there is nothing to prove. If the theorem holds for n then we prove it for $n+1$ as follows. If T is $(n+1)$-consistent and Π_{n+1}-deciding, but not Π_{n+1}-complete, there must exist some $\psi \in \Pi_{n+1}$-Th($\mathbb{N}$) such that $T \nvdash \psi$. Write $\psi = \forall z \eta(z)$ for some $\eta \in \Sigma_n$; then $\mathbb{N} \models \eta(m)$ for any $m \in \mathbb{N}$. By the induction hypothesis, T is Π_n-complete and so Σ_n-complete; thus $T \vdash \eta(\overline{m})$ for all $m \in \mathbb{N}$. On the other hand since T is Π_{n+1}-deciding and $T \nvdash \psi$ we must have $T \vdash \neg\psi$, thus $T \vdash \exists z \neg\eta(z)$. This contradicts the $(n+1)$-consistency of T. ■

Another generalization of Gödel–Rosser's Incompleteness Theorem is the following, cf. Mostowski's Theorem on *the incompleteness of any definable and ω-consistent theory* in [7, Theorem 3, p. 97]:

THEOREM 4.3 ($\overline{\Pi}_{n+1}$-incompleteness of Σ_{n+1}-definable and n-consistent theories)
If the n-consistent theory T contains **Q** and $\mathsf{Ax}_T \in \Sigma_{n+1}$, then there exists a $\overline{\Pi}_{n+1}$-sentence $\boldsymbol{\gamma}$ independent from T.

PROOF. Towards a contradiction, assume that T is $\overline{\Pi}_{n+1}$-deciding. So, T is also Π_{n+1}-deciding. Hence, by Lemma 4.2, T is Π_n-complete and so Σ_n-sound. But this contradicts Theorem 2.5. ■

Let us note that the above theorem, with a non-constructive proof, also generalizes Theorem 2.5, since n-consistency is weaker than Σ_n-soundness for $n > 2$ (cf. Proposition 3.2). And, just like before, the theorem is optimal too: The complete Σ_{n-1}-sound and Σ_{n+1}-definable theory constructed in the proof of Theorem 2.6 is also $(n-1)$-consistent by Proposition 3.2.

Our final result contains a bit of a surprise: even though the proof of Theorem 4.3 is not constructive, no one can present a constructive proof for it.

THEOREM 4.4 (Non-constructivity of n-consistency incompleteness)
Let $n \geqslant 3$ be fixed. There is no (partial) recursive function f (even with the oracle $\emptyset^{(n)}$) such that when m is a (Gödel code of a) Σ_{n+1}-formula which defines an n-consistent extension of **Q**, then $f(m)$ (halts and) is a (Gödel code of a) sentence independent from that theory.

PROOF. Assume that there is an $\emptyset^{(n)}$-(partial)recursive function f such that for any given Σ_{n+1}-formula $\Psi(x)$ if the theory $\mathcal{T}_\Psi = \{\alpha \mid \mathbb{N} \models \Psi(\ulcorner\alpha\urcorner)\}$ is an n-consistent extension of **Q** then $f(\ulcorner\Psi\urcorner)$ (halts and) is (the Gödel code of) a sentence such that $\mathcal{T}_\Psi \nvdash f(\ulcorner\Psi\urcorner)$ and $\mathcal{T}_\Psi \nvdash \neg f(\ulcorner\Psi\urcorner)$. The ω-consistency of Q with x can be written by the Π_3-formula $\boldsymbol{\omega}\text{-}\mathsf{Con}_{\mathsf{Q}}(x)$ as

$$\forall\chi\big[\exists z\mathsf{Proof}\big(z,\mathfrak{q}\wedge x\rightarrow\exists v\chi(v)\big)\rightarrow\exists v\forall z\neg\mathsf{Proof}\big(z,\mathfrak{q}\wedge x\rightarrow\neg\chi(\overline{v})\big)\big],$$

where $\mathfrak{q}$ is the Gödel code of the conjunction of the finitely many axioms of Q (see the Proofs of Theorems 2.3, 2.6). By $\emptyset^{(n)}$-(partial)recursiveness of f the expressions $y{=}f(x)$ and $f(z)\!\downarrow$ can be written by Σ_{n+1}-formulas (see e.g. [8]). By (a parametric version of) the diagonal lemma (recalling that $n\geqslant 3$) there exists some Σ_{n+1}-formula $\Theta(x)$ such that (Q proves that)

$$\begin{aligned}\Theta(x)\equiv\ &\big[f(\ulcorner\Theta\urcorner)\!\downarrow\wedge\boldsymbol{\omega}\text{-}\mathsf{Con}_{\mathsf{Q}}\big(f(\ulcorner\Theta\urcorner)\big)\wedge\big(x{=}f(\ulcorner\Theta\urcorner)\vee x{=}\mathfrak{q}\big)\big]\ \bigvee\\ &\big[f(\ulcorner\Theta\urcorner)\!\downarrow\wedge\neg\boldsymbol{\omega}\text{-}\mathsf{Con}_{\mathsf{Q}}\big(f(\ulcorner\Theta\urcorner)\big)\wedge\big(x{=}\neg f(\ulcorner\Theta\urcorner)\vee x{=}\mathfrak{q}\big)\big]\bigvee\\ &(x{=}\mathfrak{q}).\end{aligned}$$

Now, if $f(\ulcorner\Theta\urcorner)\!\uparrow$ then $\Theta(x)\equiv(x{=}\mathfrak{q})$ and so $\mathcal{T}_\Theta{=}\mathsf{Q}$ is an n-consistent extension of Q, whence $f(\ulcorner\Theta\urcorner)\!\downarrow$; contradiction. Thus, $f(\ulcorner\Theta\urcorner)\!\downarrow$. If the theory $\mathsf{Q}\cup\{f(\ulcorner\Theta\urcorner)\}$ is ω-consistent then $\Theta(x)\equiv(x{=}f(\ulcorner\Theta\urcorner)\vee x{=}\mathfrak{q})$ and so $\mathcal{T}_\Theta{=}\mathsf{Q}\cup\{f(\ulcorner\Theta\urcorner)\}$ is an n-consistent extension of Q, whence $f(\ulcorner\Theta\urcorner)$ should be independent from it; contradiction. So, $\mathsf{Q}\cup\{f(\ulcorner\Theta\urcorner)\}$ is not ω-consistent; then by [4, Theorem 21] (which states that *for any ω-consistent theory S and any sentence X either $S\cup\{X\}$ or $S\cup\{\neg X\}$ is ω-consistent*) the theory $\mathsf{Q}\cup\{\neg f(\ulcorner\Theta\urcorner)\}$ should be ω-consistent. But in this case we have $\Theta(x)\equiv(x{=}\neg f(\ulcorner\Theta\urcorner)\vee x{=}\mathfrak{q})$ and so $\mathcal{T}_\Theta{=}\mathsf{Q}\cup\{\neg f(\ulcorner\Theta\urcorner)\}$ is an n-consistent extension of Q, whence $f(\ulcorner\Theta\urcorner)$ should be independent from it; contradiction again. Thus, there can be no such $\emptyset^{(n)}$-(partial)recursive function. ∎

REMARK 4.5 (Optimality of Theorem 4.4)
There indeed exists some $\emptyset^{(n+1)}$-(total)recursive function which can find such an independent $\overline{\Pi}_{n+1}$-sentence: by having an access to the oracle $\emptyset^{(n+1)}$ for a given $\mathsf{Ax}_T\in\Sigma_{n+1}$, provability (or unprovability) in T of a given sentence is decidable. Thus (since by Theorem 4.3 there must exist some $\overline{\Pi}_{n+1}$-sentence independent from the theory T) by an exhaustive search through all the $\overline{\Pi}_{n+1}$-sentences such an independent sentence can be eventually found.

5 Conclusions

Gödel-Rosser's theorem, noting that Σ_0-soundness is equivalent to (simple) consistency in theories that contain Q, can be depicted as follows:

$$\boxed{\mathsf{Q}\subseteq T\ \&\ \mathsf{Ax}_T\in\Sigma_1\ \&\ T \text{ is } \Sigma_0\text{−Sound} \quad\Longrightarrow\quad T\not\in\Pi_1\text{−Deciding}}$$

which was generalized in Theorem 2.5 as:

$$\boxed{\mathsf{Q}\subseteq T\ \&\ \mathsf{Ax}_T\in\Sigma_{n+1}\ \&\ T \text{ is } \Sigma_n\text{−Sound} \quad\Longrightarrow\quad T\not\in\overline{\Pi}_{n+1}\text{−Deciding}}$$

with a constructive proof, and its optimality was shown in Theorem 2.6 as:

$$\boxed{\mathsf{Q}\subseteq T\ \&\ \mathsf{Ax}_T\in\Sigma_{n+1}\ \&\ T \text{ is } \Sigma_{n-1}\text{−Sound} \quad\not\Longrightarrow\quad T\not\in\text{Complete}}$$

Another form of Gödel–Rosser's Theorem is as follows:

$$\boxed{\mathsf{Q}\subseteq T\ \&\ \mathsf{Ax}_T\in\Sigma_1\ \&\ T \text{ is } 0\text{−Consistent} \quad\Longrightarrow\quad T\not\in\Pi_1\text{−Deciding}}$$

which was generalized in Theorem 4.3 as:

$$\boxed{\mathsf{Q}\subseteq T\ \&\ \mathsf{Ax}_T\in\Sigma_{n+1}\ \&\ T \text{ is } n\text{−Consistent} \quad\Longrightarrow\quad T\not\in\overline{\Pi}_{n+1}\text{−Deciding}}$$

with a non-constructive proof. This theorem is optimal too:

$$\boxed{\mathsf{Q}\subseteq T \ \& \ \mathsf{Ax}_T \in \Sigma_{n+1} \ \& \ T \text{ is } (n-1)\text{−Consistent} \quad \not\Longrightarrow \quad T \notin \text{Complete}}$$

and it was shown in Theorem 4.4 that there can be no constructive proof for it (when $n > 2$).

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