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Intuitionistic axiomatizations for bounded extension Kripke models

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Abstract

We present axiom systems, and provide soundness and strong completeness theorems, for classes of Kripke models with restricted extension rules among the node structures of the model. As examples we present an axiom system for the class of cofinal extension Kripke models, and an axiom system for the class of end-extension Kripke models. We also show that Heyting arithmetic (HA) is strongly complete for its class of end-extension models. Cofinal extension models of HA are models of Peano arithmetic (PA).

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1. Introduction

Intuitionistic predicate calculus (IQC) is sound and complete for the class of all Kripke models; see [8]. IQC is even strongly complete, that is, additionally each theory extending IQC is complete for the subclass of Kripke models for which it is sound. There has long been interest in further completeness results for subclasses of Kripke models. Early examples include a strong completeness theorem of IQC for the subclass of all Kripke models for which the underlying poset is a rooted tree of height ω. The

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following example is due to Sabine Görnemann and Dieter Klemke; see [2,4]. Let CD be the axiom schema

$$\forall x(A \vee B) \rightarrow A \vee \forall xB$$

in which x is not free in A. Görnemann and Klemke showed that CD axiomatizes a theory which is strongly complete for the subclass of Kripke models with constant domains. Our main theorem generalizes this result. We state strong completeness theorems for classes of Kripke models satisfying restrictions on what kinds of extensions of node structures are allowed 'above' other node structures in the Kripke model.

Rather than giving abstract descriptions of the full generalization, let us first consider special cases which more clearly connect with the case for constant domain models. Let $G(x,z)$ be a formula over the first-order language. Let $\mathscr{K}$ be a Kripke model. We write Dk for the domain of the node structure at node k. If $k \preccurlyeq m$, then $Dk \subseteq Dm$. We say that $\mathscr{K}$ is a G-expansion if for all $k \preccurlyeq m$, all $a \in Dk$, and all $b \in Dm$, we have $m \Vdash G(b,a)$ or $b \in Dk$. If $G(x,z)$ equals $\perp$, then $\mathscr{K}$ is a constant domain model. If $G(x,z)$ equals $\neg x \leqslant y$, then $\mathscr{K}$ is an end-extension model. Let GE be the least set of pairs of formulas such that

$$(B \vee G(x,z), \forall x(B \vee G(x,z))) \in \mathrm{GE},$$

$$(B_1,B_2) \in \mathrm{GE} \text{ implies } (A \vee B_1, A \vee B_2) \in \mathrm{GE},$$

$$(B_1,B_2) \in \mathrm{GE} \text{ implies } (A \rightarrow B_1, A \rightarrow B_2) \in \mathrm{GE},$$

$$(B_1,B_2) \in \mathrm{GE} \text{ implies } (\forall yB_1, \forall yB_2) \in \mathrm{GE}$$

for all formulas A and B, where x and z are not free in A. Let $\Sigma(\mathrm{GE})$ be the theory axiomatized by $\forall xB_1 \rightarrow B_2$, for all $(B_1,B_2) \in \mathrm{GE}$. Then $\Sigma(\mathrm{GE})$ is strongly complete for the class of G-expansion Kripke models. When we set $G(x,z)$ equal to $\neg x \leqslant y$ over the language of Heyting arithmetic (HA), then $\Sigma(\mathrm{GE}) \subseteq \mathrm{HA}$. So HA is strongly complete for its end-extension Kripke models. This answers a question posed by Kai Wehmeier; see [9].

Our main theorem also includes the example of cofinal extension Kripke models below. Let $H(x,y)$ be a formula over the first-order language. We say that a Kripke model $\mathscr{K}$ is a cofinal extension model (relative to H) if for all $k \preccurlyeq m$ and $b \in Dm$, there exist $a \in Dk$ such that $m \Vdash H(b,a)$. Let CE be the least set of pairs of formulas such that

$$(\forall y(H(y,x) \rightarrow B), \forall yB) \in \mathrm{CE},$$

$$(B_1,B_2) \in \mathrm{CE} \text{ implies } (A \vee B_1, A \vee B_2) \in \mathrm{CE},$$

$$(B_1,B_2) \in \mathrm{CE} \text{ implies } (A \rightarrow B_1, A \rightarrow B_2) \in \mathrm{CE},$$

$$(B_1,B_2) \in \mathrm{CE} \text{ implies } (\forall yB_1, \forall yB_2) \in \mathrm{CE}$$

for all formulas A and B, where x is not free in A or B. Let $\Sigma(\mathrm{CE})$ be the theory axiomatized by $\forall xB_1 \rightarrow B_2$, for all $(B_1,B_2) \in \mathrm{CE}$. Then $\Sigma(\mathrm{CE})$ is strongly complete for the class of cofinal extension Kripke models.

In general, the main theorem uses special sets of pairs of formulas R, called Z-open x-ready sets, and corresponding theories $\Sigma(R)$ axiomatized by $\forall x B_1 \to B_2$, for all $(B_1, B_2) \in R$. A Kripke model is called an R-bounded extension model if for all nodes k and all pairs $(B_1(x, \mathbf{y}, \mathbf{z}), B_2(\mathbf{y}, \mathbf{z})) \in R$ (with the notation further explained in the main text below),

$$k \Vdash \forall \mathbf{y} \bigwedge(\mathbf{d} \in Dk) \left[\bigwedge(d \in Dk) B_1(d, \mathbf{y}, \mathbf{d}) \to B_2(\mathbf{y}, \mathbf{d}) \right].$$

Then $\Sigma(R)$ is strongly complete for the class of R-bounded extension Kripke models. As is often done with completeness theorems over classes of Kripke models, we can add 'standard' refinements: If the language is countable, then $\Sigma(R)$ is strongly complete for the class of R-bounded extension Kripke models over rooted trees of height ω, such that for all $k \preccurlyeq m$ and sentences $\exists x B(x)$ over $\mathscr{L}[Dm]$, either $m \Vdash B(d)$ for some $d \in Dk$, or there exists a sentence C over $\mathscr{L}[Dm]$ such that $m \nVdash C$, and $m \Vdash B(d) \to C$ for all $d \in Dk$.

2. Bounded extension models

Kripke models over a language $\mathscr{L}$ are defined in the following standard way; see [8]:

Definition 2.1. A *Kripke model* $\mathscr{K}$ is a quadruple $(K, \preccurlyeq, D, \Vdash)$, where $(K, \preccurlyeq)$ is a nonempty partially ordered set, D is a mapping from K assigning nonempty subsets Dk to all $k \in K$ such that $k \preccurlyeq k'$ implies $Dk \subseteq Dk'$, for all $k, k' \in K$; and $\Vdash$ is the usual forcing relation between K and the set of formulas in the first-order language $\mathscr{L}$ extended with constant symbols for the elements of the corresponding sets Dk.

Above each node k we have a classical model over $\mathscr{L}$ with domain Dk. Because of the set inclusion condition $Dk \subseteq Dk'$ in the definition above, the standard equality predicate is interpreted as a congruence in these classical node models.

We extend the language $\mathscr{L}$ by adding, for each set of constant symbols D, a new 'quantifier' $\bigwedge(x \in D)A(x)$. We usually write $\bigwedge(d \in D)A(d)$ so as to distinguish it better from the familiar universal quantification $\forall x A(x)$. We extend the usual rules for the Kripke model forcing relation $\Vdash$ to this extended language by specifying, for all nodes k of a Kripke model, all sets of constant symbols D over $\mathscr{L}[Dk]$, and all formulas $A(x)$ with x as only free variable,

$$k \Vdash \bigwedge(d \in D)A(d) \text{ if and only if } k \Vdash A(d) \text{ for all } d \in D.$$

The forcing relation $\Vdash$ is extended to all formulas in the usual way. Analogously to the case for universal quantifiers, we write

$$\bigwedge(\mathbf{d} \in D)A(\mathbf{d})$$

as short for

$$\bigwedge(d_1 \in D)\bigwedge(d_2 \in D)\cdots\bigwedge(d_n \in D)A(d_1,d_2,\ldots,d_n).$$

We are not interested in extending the entailment relation $\vdash$ to the new language. We only introduce, for each set of formulas $\Sigma \cup \{A(x)\}$ over $\mathscr{L}$, the abbreviation

$$\Sigma \vdash \bigwedge(\mathbf{d} \in D)A(\mathbf{d})$$

as short for $\Sigma \vdash A(\mathbf{d})$, for all $\mathbf{d} \in D$.

Proposition 2.2. *Let A, $B(x)$, $B_1(x)$, $B_2(x)$, and $C(x,y)$ be formulas such that x is not free in A.*

1. *If* $k' \geqslant k \Vdash \bigwedge(d \in D)B(d)$, *then* $k' \Vdash \bigwedge(d \in D)B(d)$;
2. *if* $D \subseteq E$, *then* $k \Vdash \bigwedge(e \in E)B(e) \to \bigwedge(d \in D)B(d)$;
3. $k \Vdash \forall x B(x) \to \bigwedge(d \in D)B(d)$;
4. $k \Vdash \forall y \bigwedge(d \in D)C(d,y) \leftrightarrow \bigwedge(d \in D)\forall y C(d,y)$;
5. $k \Vdash \bigwedge(e \in E)\bigwedge(d \in D)C(d,e) \leftrightarrow \bigwedge(d \in D)\bigwedge(e \in E)C(d,e)$;
6. $k \Vdash \bigwedge(d \in D)(B_1(d) \wedge B_2(d)) \leftrightarrow \bigwedge(d \in D)B_1(d) \wedge \bigwedge(d \in D)B_2(d)$;
7. $k \Vdash \bigwedge(d \in D)(A \to B(d)) \leftrightarrow (A \to \bigwedge(d \in D)B(d))$; *and*
8. $k \Vdash \bigwedge(d \in D)(A \vee B(d)) \leftrightarrow (A \vee \bigwedge(d \in D)B(d))$.

Proof. We only verify two representative cases.

Case 4: We may assume $C(d,e)$ to be a sentence. Let $k \leqslant k'$. The following are equivalent:

$k' \Vdash \forall y \bigwedge(d \in D)C(d,y)$.

For all $k'' \geqslant k'$ and $e \in Dk''$, $k'' \Vdash \bigwedge(d \in D)C(d,e)$.

For all $k'' \geqslant k'$, $e \in Dk''$, and $d \in D$, $k'' \Vdash C(d,e)$.

For all $d \in D$, $k'' \geqslant k'$, and $e \in Dk''$, $k'' \Vdash C(d,e)$.

For all $d \in D$, $k' \Vdash \forall y C(d,y)$.

$k' \Vdash \bigwedge(d \in D)\forall y C(d,y)$.

Case 8: We may assume $A \vee B(d)$ to be a sentence. Let $k \leqslant k'$. The following are equivalent:

$k' \Vdash \bigwedge(d \in D)(A \vee B(d))$.

For all $d \in D$, $k' \Vdash A \vee B(d)$.

For all $d \in D$, $k' \Vdash A$ or $k' \Vdash B(d)$.

$k' \Vdash A$ or for all $d \in D$, $k' \Vdash B(d)$.

$k' \Vdash A$ or $k' \Vdash \bigwedge(d \in D)B(d)$.

$k' \Vdash A \vee \bigwedge(d \in D)B(d)$. □

Note that the metalogic of Kripke model theory is classical logic. Cases 1–7 are straightforward analogs of tautologies for standard intuitionistic universal quantification. For example, Case 7 corresponds with

$$k \Vdash \forall x(A \to B(x)) \leftrightarrow (A \to \forall x B(x)).$$

An x-*ready pair* is a pair (B_1, B_2) of formulas B_1, B_2 over $\mathscr{L}$ such that x is not free in B_2. Let Z be a set of variables with $x \notin Z$, and such that the set Y of remaining variables is still infinite. A set R is called a Z-*open* x-*ready set* over $\mathscr{L}$ if it is a set of x-ready pairs of formulas over $\mathscr{L}$, closed under the operations

- If $(B_1, B_2) \in R$ and A is a formula over $\mathscr{L}$ in which none of the variables in $Z \cup \{x\}$ is free, then $(A \to B_1, A \to B_2) \in R$;
- if $(B_1, B_2) \in R$ and A is a formula over $\mathscr{L}$ in which none of the variables in $Z \cup \{x\}$ is free, then $(A \vee B_1, A \vee B_2) \in R$; and
- if $(B_1, B_2) \in R$ and y is a variable not in $Z \cup \{x\}$, then $(\forall y B_1, \forall y B_2) \in R$.

R is called a *closed* x-*ready set* over $\mathscr{L}$ when it is an $\emptyset$-open x-ready set over $\mathscr{L}$.

As a corollary to Proposition 2.2 we get

Proposition 2.3. *Let* $(B_1(x), B_2)$ *be an* x*-ready pair of formulas over* $\mathscr{L}[Dk]$, *and* D *a set of constant symbols, such that*

$$k \Vdash \bigwedge(d \in D) B_1(d) \to B_2.$$

If A *is a formula over* $\mathscr{L}[Dk]$ *with no free occurrences of* x, *then*

$$k \Vdash \bigwedge(d \in D)(A \to B_1(d)) \to (A \to B_2),$$

and

$$k \Vdash \bigwedge(d \in D)(A \vee B_1(d)) \to (A \vee B_2).$$

If y *is a variable different from* x, *then*

$$k \Vdash \bigwedge(d \in D)\forall y B_1(d) \to \forall y B_2.$$

So the collection $\{(B_1(x), B_2) \mid k \Vdash \bigwedge(d \in D) B_1(d) \to B_2\}$ *forms a closed* x*-ready set over* $\mathscr{L}[Dk]$.

Proof. Use Proposition 2.2, Cases 7, 8, and 4. $\square$

Each element of a Z-open x-ready set R can be written as $(B_1(x, \mathbf{y}, \mathbf{z}), B_2(\mathbf{y}, \mathbf{z}))$, where $\mathbf{z}$ lists all free variables from Z that occur in the pair, and $\mathbf{y}$ lists all remaining free variables, minus x. To simplify notations, we sometimes refer to R without explicitly specifying Z or x. For convenience we may think of Z and x as fixed throughout Section 2. A Kripke model over $\mathscr{L}$ is an R-*bounded extension model* if for all nodes k, and all $(B_1(x,\mathbf{y},\mathbf{z}),B_2(\mathbf{y},\mathbf{z}))\in R$,

$$k \Vdash \forall \mathbf{y} \bigwedge(\mathbf{d} \in Dk)\left[\bigwedge(d \in Dk)B_1(d,\mathbf{y},\mathbf{d}) \to B_2(\mathbf{y},\mathbf{d})\right].$$

Given a Z-open x-ready set R over $\mathscr{L}$, let $\Sigma(R)$ be the theory over $\mathscr{L}$ axiomatized by $\{\forall x B_1 \to B_2 \mid (B_1,B_2)\in R\}$, where we identify formulas with their universal closures in the usual way.

Lemma 2.4. *Let k be a node of a Kripke model, D a set of constant symbols over $\mathscr{L}[Dk]$, and $(B_1(x),B_2)$ an x-ready pair of formulas over $\mathscr{L}[Dk]$, such that $k \Vdash \bigwedge(d\in D)B_1(d)\to B_2$. Then $k\Vdash \forall x B_1(x)\to B_2$.*

Proof. Immediate from Proposition 2.2, Case 3. □

Proposition 2.5 (Soundness). *Let R be a Z-open x-ready set, and $\mathscr{K}$ a Kripke model such that $\mathscr{K}$ is an R-bounded extension model. Then $\mathscr{K}\models\Sigma(R)$.*

Proof. Let $(B_1(x,\mathbf{z}),B_2(\mathbf{z}))\in R$, with all free variables from Z displayed. So for all nodes k we have $k\Vdash\bigwedge(\mathbf{d}\in Dk)(\bigwedge(d\in Dk)B_1(d,\mathbf{d})\to B_2(\mathbf{d}))$. Now $k\Vdash\forall x B_1(x,\mathbf{z})\to B_2(\mathbf{z})$, if and only if for all $k'\geqslant k$, and all $\mathbf{d}'\in Dk'$, $k'\Vdash\forall x B_1(x,\mathbf{d}')\to B_2(\mathbf{d}')$. By assumption, $k'\Vdash\bigwedge(d\in Dk')B_1(d,\mathbf{d}')\to B_2(\mathbf{d}')$. Apply Lemma 2.4. □

Strong completeness, the 'reverse' of Proposition 2.5, holds too. In the completeness theorem of this section, we want our models to satisfy certain extra properties, standard refinements as mentioned at the end of the introduction, which we use in the examples. Consequently, we only state a result for countable languages. The completeness theorem without cardinality restrictions will be discussed in Section 4.

Lemma 2.6. *Let Σ be a set of sentences, and D a set of constant symbols, infinitely many of which do not occur in Σ. Then for all x-ready pairs $(B_1(x),B_2)$, if $\Sigma\vdash\forall x B_1(x)\to B_2$ and $\Sigma\vdash\bigwedge(d\in D)B_1(d)$, then $\Sigma\vdash B_2$.*

Proof. If $\Sigma\vdash\bigwedge(d\in D)B_1(d)$, then $\Sigma\vdash B_1(d)$ for some d which does not occur in $\Sigma\cup\{B_1(x)\}$. So $\Sigma\vdash\forall x B_1(x)$. □

Lemma 2.7. *Let $\Sigma\cup\{A,B,C\}$ be a set of sentences. If $\Sigma\cup\{A\vee B\}\nvdash C$, then $\Sigma\cup\{A\}\nvdash C$ or $\Sigma\cup\{B\}\nvdash C$.*

Let $\Sigma\cup\{\exists x A(x),B\}$ be a set of sentences, and d a constant symbol that does not occur in this set. If $\Sigma\cup\{\exists x A(x)\}\nvdash B$, then $\Sigma\cup\{A(d)\}\nvdash B$.

Proof. Standard. □

Lemma 2.8. *Let R be a closed x-ready set, $\Sigma\cup\{A,B\}$ a set of sentences, and D a set of constant symbols. Suppose $\Sigma\vdash\bigwedge(d\in D)B_1(d)$ implies $\Sigma\vdash B_2$, for all $(B_1(x),B_2)\in R$. Then if $(B_1(x),B_2)\in R$ is such that $\Sigma\cup\{A\}\nvdash B_2\vee B$, then there is $d\in D$ such that $\Sigma\cup\{A\}\nvdash B_1(d)\vee B$.*

Proof. Suppose $(B_1(x), B_2) \in R$ is such that $\Sigma \cup \{A\} \vdash B_1(d) \vee B$, for all $d \in D$. So $\Sigma \vdash \bigwedge(d \in D)(A \to (B_1(d) \vee B))$. Now $(A \to (B_1(x) \vee B), A \to (B_2 \vee B)) \in R$, so $\Sigma \vdash A \to (B_2 \vee B)$. Thus $\Sigma \cup \{A\} \vdash B_2 \vee B$. □

Recall that a theory Σ is saturated when (1) $\Sigma \vdash A \vee B$ implies $\Sigma \vdash A$ or $\Sigma \vdash B$, for all sentences $A \vee B$; and (2) $\Sigma \vdash \exists x A(x)$ implies $\Sigma \vdash A(d)$ for some constant symbol d, for all sentences $\exists x A(x)$.

Lemma 2.9. *Let $\mathscr{L}$ be a countable language, let $\{R_i\}_{i<\omega}$ be a collection of closed x-ready sets, and $\Sigma \cup \{A\}$ be a set of sentences. Let $\{D_i\}_{i<\omega} \cup \{D\}$ be a collection of nonempty sets of constant symbols. Suppose that*

- $\Sigma \nvdash A$;
- *infinitely many constants of D do not occur in Σ; and*
- *for all i, if $(B_1(x), B_2) \in R_i$, and $\Sigma \vdash \bigwedge(d \in D_i) B_1(d)$, then $\Sigma \vdash B_2$.*

Then there is a saturated theory $\Sigma' \supseteq \Sigma$ such that

- $\Sigma' \nvdash A$;
- *for all i, if $(B_1(x), B_2) \in R_i$, and $\Sigma' \vdash \bigwedge(d \in D_i) B_1(d)$, then $\Sigma' \vdash B_2$; and*
- *for all i and all sentences $\exists x B(x)$, if $\Sigma' \nvdash B(d)$ for all $d \in D_i$, then there exists a sentence C such that*
  - $\Sigma' \nvdash C$; *and*
  - $\Sigma' \vdash \bigwedge(d \in D_i)(B(d) \to C)$.

Proof. We construct a sequence $(\Gamma_0, \Delta_0), (\Gamma_1, \Delta_1), (\Gamma_2, \Delta_2), \ldots$ of pairs of finite sets of sentences such that for all i, $\Gamma_i \subseteq \Gamma_{i+1}$, $\Delta_i \subseteq \Delta_{i+1}$, and $\Sigma \cup \Gamma_i \nvdash \bigvee \Delta_i$. There are countably many sentences of the form $B_1 \vee B_2$, countably many pairs $(\exists x B(x), n)$ of existential sentences and integers, and countably many triples $(B_1(x), B_2, n)$ such that $(B_1(x), B_2) \in R_n$ and x is the only free variable. Let $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ be an enumeration of all these disjunctions, pairs, and triples. Construct the pairs (Γ_i, Δ_i) as follows:

- Set $\Gamma_0 = \emptyset$ and $\Delta_0 = \{A\}$.
- Suppose (Γ_i, Δ_i) has been constructed, and ε_i is of the form $B_1 \vee B_2$. Set $\Delta_{i+1} = \Delta_i$. If possible, set $\Gamma_{i+1} = \Gamma_i \cup \{B_j\}$ for some j such that $\Sigma \cup \Gamma_{i+1} \nvdash \bigvee \Delta_{i+1}$. Otherwise, by Lemma 2.7, $\Sigma \cup \Gamma_i \cup \{B_1 \vee B_2\} \vdash \bigvee \Delta_{i+1}$, and thus $\Sigma \cup \Gamma_i \nvdash B_1 \vee B_2$. Set $\Gamma_{i+1} = \Gamma_i$.
- Suppose (Γ_i, Δ_i) has been constructed, and ε_i is of the form $(\exists x B(x), n)$. Set $\Delta_{i+1} = \Delta_i$. If $\Sigma \cup \Gamma_i \cup \{B(d)\} \nvdash \bigvee \Delta_{i+1}$ for some $d \in D_n$, set $\Gamma_{i+1} = \Gamma_i \cup \{B(d)\}$. Otherwise, if $\Sigma \cup \Gamma_i \cup \{B(d)\} \nvdash \bigvee \Delta_{i+1}$ for some $d \in D$, set $\Gamma_{i+1} = \Gamma_i \cup \{B(d)\}$. If that is not possible either then, by Lemma 2.7, $\Sigma \cup \Gamma_i \nvdash \exists x B(x)$. Set $\Gamma_{i+1} = \Gamma_i$.
- Suppose (Γ_i, Δ_i) has been constructed, and ε_i is of the form $(B_1(x), B_2, n)$. Set $\Gamma_{i+1} = \Gamma_i$. If $\Sigma \cup \Gamma_{i+1} \vdash B_2 \vee \bigvee \Delta_i$, set $\Delta_{i+1} = \Delta_i$. Otherwise, by Lemma 2.8 there is $d \in D_n$ such that $\Sigma \cup \Gamma_{i+1} \nvdash B_1(d) \vee \bigvee \Delta_i$. Set $\Delta_{i+1} = \Delta_i \cup \{B_1(d)\}$.

Set $\Sigma' = \Sigma \cup \bigcup_i \Gamma_i$. Clearly $\Sigma' \nvdash B$, for all $B \in \bigcup_i \Delta_i$. In particular, $\Sigma' \nvdash A$. Suppose $\Sigma' \vdash B_1 \vee B_2$. There is i such that ε_i equals $B_1 \vee B_2$. Then $\Sigma \cup \Gamma_i \cup \{B_1 \vee B_2\} \nvdash \bigvee \Delta_{i+1}$.

So $B_1 \in \Gamma_{i+1}$ or $B_2 \in \Gamma_{i+1}$. Thus $\Sigma' \vdash B_1$ or $\Sigma' \vdash B_2$. Similarly, if $\Sigma' \vdash \exists x B(x)$, then $B(d) \in \Sigma'$ for some constant symbol d. So Σ' is a saturated theory. Suppose $(B_1(x, \mathbf{y}), B_2(\mathbf{y})) \in R_n$, where $\mathbf{y}$ includes all free variables different from x, is such that $\Sigma' \nvdash B_2(\mathbf{y})$. There is i such that ε_i equals $(\forall \mathbf{y} B_1(x, \mathbf{y}), \forall \mathbf{y} B_2(\mathbf{y}), n)$. If $\Sigma \cup \Gamma_{i+1} \vdash \forall \mathbf{y} B_2(\mathbf{y}) \vee \bigvee \Delta_i$ then, by saturatedness, $\Sigma' \vdash \forall \mathbf{y} B_2(\mathbf{y})$, contradicting our assumption. So $\Sigma \cup \Gamma_{i+1} \nvdash \forall \mathbf{y} B_2(\mathbf{y}) \vee \bigvee \Delta_i$. Then there is $d \in D_n$ such that $\forall \mathbf{y} B_1(d, \mathbf{y}) \in \Delta_{i+1}$. So $\Sigma' \nvdash B_1(d, \mathbf{y})$. Finally, suppose $\exists x B(x)$ is a sentence, and n an integer, such that $\Sigma' \nvdash B(d)$, for all $d \in D_n$. There is i such that ε_i equals $(\exists x B(x), n)$. Then $\Sigma \cup \Gamma_i \vdash \bigwedge (d \in D_n)(B(d) \rightarrow \bigvee \Delta_{i+1})$. □

We call a preordered set $(K, \preccurlyeq)$ a *tree* if $\preccurlyeq$ is a partial order such that there is a least element, and such that the predecessors of each element form a finite set, linearly ordered by $\preccurlyeq$. A Kripke model is called a *tree model* if its preordered underlying set of nodes is a tree.

Theorem 2.10 (Countable completeness). *Let R be a Z-open x-ready set over a countable language $\mathscr{L}$, and let $\Sigma \cup \{A\}$ be a set of sentences of $\mathscr{L}$ such that*

- $\Sigma \supseteq \Sigma(R)$; *and*
- $\Sigma \nvdash A$.

Then there exists a Kripke tree model $\mathscr{K}$ of Σ such that $\mathscr{K} \nVdash A$, and such that $\mathscr{K}$ satisfies the following conditions:

- *$\mathscr{K}$ is an R-bounded extension model*; *and*
- *for all nodes $k \preccurlyeq k'$ and sentences $\exists x B(x)$ over $\mathscr{L}[Dk']$, if for all $d \in Dk$, $k' \nVdash B(d)$, then there exists a sentence C over $\mathscr{L}[Dk']$ such that*
  - *$k' \nVdash C$; and*
  - *$k' \Vdash \bigwedge (d \in Dk)(B(d) \rightarrow C)$.*

Proof. Let $D_0 \subseteq D_1 \subseteq D_2 \subseteq \cdots$ be sets of constant symbols such that D_0 is the set of all constant symbols of $\mathscr{L}$, and such that $D_{i+1} \backslash D_i$ is countably infinite, for all i. Let $R[D_i]$ denote the smallest Z-open x-ready set over $\mathscr{L}[D_i]$ containing R. So each pair of $R[D_i]$ can be obtained from a pair $(B_1(x, \mathbf{y}, \mathbf{z}), B_2(\mathbf{y}, \mathbf{z})) \in R$ by substituting constant symbols from D_i for some of the variables from $\mathbf{y}$. As nodes of the Kripke model $\mathscr{K}$ we choose all finite sequences $k = \langle \Sigma_1, \Sigma_2, \ldots, \Sigma_m \rangle$ such that

- Σ_j is a saturated theory over $\mathscr{L}[D_j]$, for all j;
- $\Sigma \subseteq \Sigma_1$ is some fixed theory, constructed below;
- $\Sigma_j \subseteq \Sigma_{j+1}$, for all j;
- for all $i \leqslant j$, all $(B_1(x, \mathbf{z}), B_2(\mathbf{z})) \in R[D_j]$ where $\mathbf{z}$ lists all variables from Z, and all $\mathbf{d} \in D_i$, if $\Sigma_j \vdash \bigwedge (d \in D_i) B_1(d, \mathbf{d})$, then $\Sigma_j \vdash B_2(\mathbf{d})$; and
- for all $i \leqslant j$, and for each sentence $\exists x B(x)$ over $\mathscr{L}[D_j]$, if $\Sigma_j \nvdash B(d)$ for all $d \in D_i$, then there is a sentence C over $\mathscr{L}[D_j]$ such that
  - $\Sigma_j \nvdash C$; and
  - $\Sigma_j \vdash \bigwedge (d \in D_i)(B(d) \rightarrow C)$.

Set $k \preccurlyeq k'$ exactly when k is an initial segment of k'. Set $Dk = D_m$. Set $k \Vdash B$ for atomic sentences B over $\mathscr{L}[D_m]$, exactly when $\Sigma_m \vdash B$. The structure thus defined is a Kripke tree model if the set of nodes is nonempty.

Next we construct a node. The set D_1 contains infinitely many constants that are not in Σ. So, by Lemma 2.6, if $(B_1(x, \mathbf{z}), B_2(\mathbf{z})) \in R[D_1]$ with $\mathbf{z}$ all free variables from Z, and $\mathbf{d} \in D_1$ are such that $\Sigma \vdash \bigwedge(d \in D_1) B_1(d, \mathbf{d})$, then $\Sigma \vdash B_2(\mathbf{d})$. The set of all x-ready pairs formed from pairs of $R[D_1]$ by substituting constant symbols of D_1 for all the Z variables, forms a closed x-ready set over $\mathscr{L}[D_1]$. By Lemma 2.9 there exists a saturated theory Σ_1 over $\mathscr{L}[D_1]$ such that

- $\Sigma \subseteq \Sigma_1 \nvdash A$;
- $\Sigma_1 \vdash \bigwedge(d \in D_1) B_1(d, \mathbf{d})$ implies $\Sigma_1 \vdash B_2(\mathbf{d})$, for all $(B_1(x, \mathbf{z}), B_2(\mathbf{z})) \in R[D_1]$ with $\mathbf{z}$ all free variables from Z that occur in B_1 or in B_2, and all $\mathbf{d} \in D_1$; and
- for each sentence $\exists x B(x)$ over $\mathscr{L}[D_1]$, if $\Sigma_1 \nvdash B(d)$ for all $d \in D_1$, then there is a sentence C over $\mathscr{L}[D_1]$ such that
  - $\Sigma_1 \nvdash C$; and
  - $\Sigma_1 \vdash \bigwedge(d \in D_1)(B(d) \to C)$.

So $\langle \Sigma_1 \rangle$ is a node such that $\Sigma_1 \nvdash A$.

Next we show that for all nodes $k = \langle \Sigma_1, \Sigma_2, \ldots, \Sigma_m \rangle$, and all sentences B over $\mathscr{L}[D_m]$, $k \Vdash B$ if and only if $\Sigma_m \vdash B$. We prove the claim by induction on the complexity of B. By definition the case holds for atomic sentences, and is preserved under conjunction and, because of saturation, under disjunction and existential quantification. Implication: Suppose $\Sigma_m \vdash C_1 \to C_2$, and $k \preccurlyeq k' \Vdash C_1$, where k' has length $n \geqslant m$, and theory Σ_n is the last theory in the sequence k'. So $\Sigma_n \supseteq \Sigma_m$. By induction, $\Sigma_n \vdash C_1$, so $\Sigma_n \vdash C_2$. Again by induction, $k' \Vdash C_2$. Thus $k \Vdash C_1 \to C_2$. Conversely, suppose $\Sigma_m \nvdash C_1 \to C_2$. Set $\Sigma_+ = \Sigma_m \cup \{C_1\}$. Then $\Sigma_+ \nvdash C_2$. Let $(B_1(x, \mathbf{z}), B_2(\mathbf{z})) \in R[D_{m+1}]$, where $\mathbf{z}$ lists all free variables from Z. Suppose $i \leqslant m$ and $\mathbf{d} \in D_i$ such that $\Sigma_+ \vdash \bigwedge(d \in D_i) B_1(d, \mathbf{d})$ or, equivalently, $\Sigma_m \vdash \bigwedge(d \in D_i)(C_1 \to B_1(d, \mathbf{d}))$. There are a pair $(B'_1(x, \mathbf{y}, \mathbf{z}), B'_2(\mathbf{y}, \mathbf{z})) \in R[D_m]$ and constant symbols $\mathbf{d}' \in D_{m+1} \backslash D_m$ such that $(B'_1(x, \mathbf{d}', \mathbf{z}), B'_2(\mathbf{d}', \mathbf{z}))$ equals $(B_1(x, \mathbf{z}), B_2(\mathbf{z}))$. Then $\Sigma_m \vdash \bigwedge(d \in D_i)(C_1 \to B'_1(d, \mathbf{y}, \mathbf{d}))$, and thus $\Sigma_m \vdash C_1 \to B'_2(\mathbf{y}, \mathbf{d})$. So also $\Sigma_+ \vdash B_2(\mathbf{d})$. Suppose $i = m+1$ and $\mathbf{d} \in D_{m+1}$ such that $\Sigma_+ \vdash \bigwedge(d \in D_{m+1}) B_1(d, \mathbf{d})$. Then, by Lemma 2.6, $\Sigma_+ \vdash B_2(\mathbf{d})$. For all $i \leqslant m+1$ the set of all x-ready pairs formed from pairs of $R[D_{m+1}]$ by substituting constant symbols of D_i for all the Z variables, forms a closed x-ready set over $\mathscr{L}[D_{m+1}]$. By Lemma 2.9 there exists a saturated theory Σ_{m+1} over $\mathscr{L}[D_{m+1}]$ such that

- $\Sigma_+ \subseteq \Sigma_{m+1} \nvdash C_2$;
- for all $i \leqslant m+1$, all $(B_1(x, \mathbf{z}), B_2(\mathbf{z})) \in R[D_{m+1}]$ where $\mathbf{z}$ lists all variables from Z, and all $\mathbf{d} \in D_i$, if $\Sigma_{m+1} \vdash \bigwedge(d \in D_i) B_1(d, \mathbf{d})$, then $\Sigma_{m+1} \vdash B_2(\mathbf{d})$; and
- for all $i \leqslant m+1$, and for each sentence $\exists x B(x)$ over $\mathscr{L}[D_{m+1}]$, if $\Sigma_{m+1} \nvdash B(d)$ for all $d \in D_i$, then there is a sentence C over $\mathscr{L}[D_{m+1}]$ such that
  - $\Sigma_{m+1} \nvdash C$; and
  - $\Sigma_{m+1} \vdash \bigwedge(d \in D_i)(B(d) \to C)$.

Then $k \preccurlyeq k' = \langle \Sigma_1, \Sigma_2, \ldots, \Sigma_m, \Sigma_{m+1} \rangle$ are nodes such that $k' \Vdash C_1$ and $k' \nVdash C_2$. Thus $k \nVdash C_1 \to C_2$. Universal quantification: It is an easy exercise to show the induction step that if $\Sigma_m \vdash \forall x B(x)$, then $k \Vdash \forall x B(x)$. Conversely, suppose that $\Sigma_m \nvdash \forall x B(x)$. Let Σ_+ be the theory over $\mathscr{L}[D_{m+1}]$ axiomatized by Σ_m, and let $e \in D_{m+1} \backslash D_m$. Then $\Sigma_+ \nvdash B(e)$. Analogously to the case for implication, there is a node $\langle \Sigma_1, \Sigma_2, \ldots, \Sigma_m, \Sigma_{m+1} \rangle = k' \succcurlyeq k$ such that $\Sigma_{m+1} \nvdash B(e)$. So $k' \nVdash B(e)$, and thus $k \nVdash \forall x B(x)$.

Let $(B_1(x, \mathbf{y}, \mathbf{z}), B_2(\mathbf{y}, \mathbf{z})) \in R$ with $\mathbf{z}$ all free variables from Z, and $\mathbf{y}$ all other free variables different from x, let $k = \langle \Sigma_1, \Sigma_2, \ldots, \Sigma_m \rangle$ be a node, and $\mathbf{d} \in Dk = D_m$. To show: $k \Vdash \bigwedge(d \in Dk) B_1(d, \mathbf{y}, \mathbf{d}) \to B_2(\mathbf{y}, \mathbf{d})$. Suppose $k \preccurlyeq k' = \langle \Sigma_1, \ldots, \Sigma_m, \ldots, \Sigma_n \rangle$ and $\mathbf{d}' \in Dk' = D_n$ are such that $k' \Vdash \bigwedge(d \in Dk) B_1(d, \mathbf{d}', \mathbf{d})$. Then $\Sigma_n \vdash \bigwedge(d \in Dk) B_1(d, \mathbf{d}', \mathbf{d})$, so $\Sigma_n \vdash B_2(\mathbf{d}', \mathbf{d})$. Thus $k' \Vdash B_2(\mathbf{d}', \mathbf{d})$.

The remaining property on the existence of sentences C is now straightforward. □

Modulo simple renaming of variables we may assume that all Z-open x-ready sets are chosen with a fixed (countably) infinite set Z and a fixed variable x. Then the definitions trivially imply that the collection of Z-open x-ready sets is closed under arbitrary unions and intersections. Consequently the countable completeness theorem 2.10 easily extends to collections of Z-open x-ready sets.

3. Applications

3.1. Cofinal extension models

Let the language $\mathscr{L}$ be provided with a binary predicate $x \leqslant y$.

Definition 3.1. A Kripke model $\mathscr{K}$ is a *cofinal extension model* if for all $k \preccurlyeq k'$, and $b \in Dk'$ there exist $a \in Dk$ such that $k' \Vdash b \leqslant a$.

Lemma 3.2. *Let $\mathscr{K}$ be a cofinal extension model. Then for all nodes k, and all formulas $B(y)$ over $\mathscr{L}[Dk]$,*

$$k \Vdash \bigwedge(d \in Dk) \forall y (y \leqslant d \to B(y)) \to \forall y B(y).$$

Proof. Let $k' \succcurlyeq k$. We may assume that $\forall y B(y)$ is a sentence. Suppose $k \Vdash \bigwedge(d \in Dk) \forall y (y \leqslant d \to B(y))$, and $b \in Dk'$. There exists $a \in Dk$ such that $k' \Vdash b \leqslant a$. With $k' \Vdash b \leqslant a \to B(b)$ this implies $k' \Vdash B(b)$. □

Let CE be the closed x-ready set generated by all pairs of the form

$$(\forall y (y \leqslant x \to B(y)), \forall y B(y)),$$

where x is not free in $B(y)$.

Theorem 3.3. *Let $\mathscr{L}$ be a countable language. Then the theory $\Sigma(\mathrm{CE})$ is sound and strongly complete for the class of cofinal extension Kripke models.*

Proof. Soundness immediately follows from Lemmas 3.2 and 2.4, and Proposition 2.3. For strong completeness, let $\Sigma \cup \{A\}$ be a set of sentences such that $\Sigma \supseteq \Sigma(\mathrm{CE})$ and $\Sigma \nvdash A$. There is a Kripke model $\mathscr{K}$ of Σ satisfying the conclusions of Theorem 2.10. In particular, $\mathscr{K} \nvDash A$. Let $k \preccurlyeq k'$ and $d' \in Dk'$. To show: $k' \Vdash d' \leqslant d$ for some $d \in Dk$. Suppose not. Consider the existential sentence $\exists x B(x)$ with $B(x)$ equal to $d' \leqslant x$. There is a sentence C over $\mathscr{L}[Dk']$ such that

- $k' \nVdash C$; and
- $k' \Vdash \bigwedge(d \in Dk)(d' \leqslant d \rightarrow C)$.

This implies $k' \Vdash \bigwedge(d \in Dk)\forall y(y \leqslant d \rightarrow (d' = y \rightarrow C))$. Since $\mathscr{K}$ is a CE-bounded extension model, $k' \Vdash \forall y(d' = y \rightarrow C)$. So $k' \Vdash C$, contradiction. So $k' \Vdash d' \leqslant d$ for some $d \in Dk$. □

In Section 4 we show how to remove the cardinality restriction from Theorem 3.3.

3.2. *G*-expansions and end-extensions

Let $G(x, \mathbf{z}) = G(x, z_1, \ldots, z_n)$ be a formula over $\mathscr{L}$, where $x, \mathbf{z}$ lists all free variables of G.

Definition 3.4. A Kripke model $\mathscr{K}$ is a *G-expansion* if for all $k \preccurlyeq k'$, all $\mathbf{a} = a_1, \ldots, a_n \in Dk$, and all $b \in Dk'$, we have $k' \Vdash G(b, \mathbf{a})$, or $b \in Dk$.

Obviously all models are G-expansions when G equals $\top$. When G equals $\bot$, we get the constant domain models (Grzegorczyk models).

Given a formula $F(x, \mathbf{z})$, define $\mathscr{K}$ to be a (*weak*) *F end-extension model* when $\mathscr{K}$ is a $\neg F$-expansion.

Proposition 3.5. *A model $\mathscr{K}$ is an F end-extension model, if and only if for all $k \preccurlyeq k'$, all $\mathbf{a} = a_1, \ldots, a_n \in Dk$, and all $b \in Dk'$, if $k' \Vdash F(b, \mathbf{a})$, then $b \in Dk$.*

Proof. Suppose $\mathscr{K}$ is a $\neg F$-expansion, and $k' \Vdash F(b, \mathbf{a})$, with $a_1, \ldots, a_n \in Dk$, with $k \preccurlyeq k'$, and with $b \in Dk'$. Then $k' \nVdash \neg F(b, \mathbf{a})$, so $b \in Dk$. Conversely, suppose that for all $k \preccurlyeq k'$, all $\mathbf{a} = a_1, \ldots, a_n \in Dk$, and all $b \in Dk'$, if $k' \Vdash F(b, \mathbf{a})$, then $b \in Dk$. Suppose $k' \nVdash \neg F(b, \mathbf{a})$. Then there is $k'' \succcurlyeq k'$ such that $k'' \Vdash F(b, \mathbf{a})$. So $b \in Dk$. □

Standard examples of end-extension models are special models of HA, where $F(x, y)$ is the predicate $x \leqslant y$, and special models of set theory, where $F(x, y)$ is the predicate $x \in y$.

In this section we present theories for which we prove soundness and strong completeness theorems with respect to the class of G-expansion models. This implies that we also have theories for which we establish soundness and strong completeness theorems with respect to the class of F end-extension models.

Lemma 3.6. *Let $\mathscr{K}$ be a G-expansion model, k a node, and $B(x)$ be a formula over $\mathscr{L}$. Then*

$$k \Vdash \bigwedge(\mathbf{a} \in Dk)\left[\bigwedge(d \in Dk)(B(d) \vee G(d,\mathbf{a})) \rightarrow \forall x(B(x) \vee G(x,\mathbf{a}))\right].$$

Proof. Let $k' \geqslant k$, $B(d)$ be a sentence over $\mathscr{L}[Dk']$, and $\mathbf{a} \in Dk$. It suffices to show

$$k' \Vdash \bigwedge(d \in Dk)(B(d) \vee G(d,\mathbf{a})) \rightarrow \forall x(B(x) \vee G(x,\mathbf{a})).$$

Suppose $k' \nVdash \forall x(B(x) \vee G(x,\mathbf{a}))$. Then there exist $k'' \geqslant k'$ and $d \in Dk''$ such that $k'' \nVdash B(d) \vee G(d,\mathbf{a})$. So $k'' \nVdash B(d)$, and $k'' \nVdash G(d,\mathbf{a})$. But then $d \in Dk$ with $k' \nVdash B(d)$ and $k' \nVdash G(d,\mathbf{a})$. So $d \in Dk$ such that $k' \nVdash B(d) \vee G(d,\mathbf{a})$. □

Let GE be the Z-open x-ready set generated by all pairs of the form

$$(B(x) \vee G(x,\mathbf{z}), \forall x(B(x) \vee G(x,\mathbf{z}))),$$

where Z is the set of variables in $\mathbf{z}$.

Theorem 3.7. *Let $\mathscr{L}$ be a countable language. Then the theory $\Sigma(\mathrm{GE})$ is sound and strongly complete for the class of G-expansion Kripke models.*

Proof. Soundness immediately follows from Lemmas 3.6 and 2.4, and Proposition 2.3. For strong completeness, let $\Sigma \cup \{A\}$ be a set of sentences such that $\Sigma \supseteq \Sigma(\mathrm{GE})$ and $\Sigma \nvdash A$. There is a Kripke tree model $\mathscr{K}$ of Σ satisfying the conclusions of Theorem 2.10. In particular, $\mathscr{K} \nvDash A$. We first show a slightly weaker property than G-expansion: If $k \preccurlyeq k'$, $\mathbf{a} \in Dk$, and $d' \in Dk'$, then $k' \Vdash G(d',\mathbf{a})$, or $k' \Vdash d' = b$ for some $b \in Dk$. Suppose there is no $b \in Dk$ such that $k' \Vdash d' = b$. Consider the existential sentence $\exists x B(x)$ with $B(x)$ equal to $d' = x$. There is a sentence C over $\mathscr{L}[Dk']$ such that

- $k' \nVdash C$; and
- $k' \Vdash \bigwedge(d \in Dk)(d' = d \rightarrow C)$.

So $k' \Vdash \bigwedge(d \in Dk)((d' = d \rightarrow C) \vee G(d,\mathbf{a}))$. Since $\mathscr{K}$ is a GE-bounded extension model, $k' \Vdash \forall x((d' = x \rightarrow C) \vee G(x,\mathbf{a}))$. So, in particular, $k' \Vdash C \vee G(d',\mathbf{a})$. And thus $k' \Vdash G(d',\mathbf{a})$. So $\mathscr{K}$ satisfies the slightly weaker property. Finally, we 'prune' the tree model $\mathscr{K}$ as follows: Whenever $k \preccurlyeq k'$ are immediate successor nodes, and $d' \in Dk' \backslash Dk$ is such that $k' \Vdash d' = b$ for some $b \in Dk$, then remove d' from all Dk'' with $k' \preccurlyeq k''$. This careful pruning makes that the resulting pruned tree substructure $\mathscr{K}^-$ satisfies $D^-k_1 \subseteq D^-k_2$ whenever $k_1 \preccurlyeq k_2$. So $\mathscr{K}^-$ is a Kripke model which still satisfies the conclusions of Theorem 2.10. Let $k \preccurlyeq k'$, $\mathbf{a} \in D^-k$, and $b \in D^-k'$, such that $k' \nVdash G(b,\mathbf{a})$. Among the predecessors of k' there is a first node k_1 such that $b \in D^-k_1$. If $k_1 \preccurlyeq k$, then $b \in D^-k$. Suppose not. Then $k_1 \nVdash G(b,\mathbf{a})$. So there is $a \in D^-k$ such that $k_1 \Vdash b = a$. By the pruning property, $b \in D^{-}k_2$, where k_2 is the immediate predecessor of k_1, contradicting the minimality of k_1. Thus $b \in D^{-}k$. □

In Section 4 we show how to remove the cardinality restriction from Theorem 3.7.

3.3. Applications to Heyting arithmetic

It is well-known that over Heyting arithmetic (HA) the usual formula $x \leqslant y$ is decidable, that is, $\mathrm{HA} \vdash x \leqslant y \vee \neg x \leqslant y$. Slightly less well-known is the following:

Lemma 3.8. HA *satisfies the schema*

$$\forall x(x \leqslant y \to (A(y) \vee B(x,y))) \to (A(y) \vee \forall x(x \leqslant y \to B(x,y))),$$

for all formulas A and B such that x is not free in A.

Proof. Let $C(y,z)$ be the formula.

$$\begin{aligned}&\forall x(x \leqslant y \to [A(y+z) \vee B(x,y+z)])\\&\quad \to (A(y+z) \vee \forall x(x \leqslant y \to B(x,y+z))).\end{aligned}$$

Obviously, $\mathrm{HA} \vdash \forall z C(0,z)$.

$$\begin{aligned}\mathrm{HA} \vdash C(y,z+1) \wedge \forall x(x \leqslant (y+1) \to [A(y+1+z) \vee B(x,y+1+z)])&\\ \to \forall x(x \leqslant y \to [A(y+1+z) \vee B(x,y+1+z)])&\\ \quad \wedge [A(y+1+z) \vee B(y+1,y+1+z)]&\\ \to [A(y+1+z) \vee \forall x(x \leqslant y \to B(x,y+1+z))]&\\ \quad \wedge [A(y+1+z) \vee B(y+1,y+1+z)]&\\ \to A(y+1+z) \vee [\forall x(x \leqslant y \to B(x,y+1+z)) \wedge B(y+1,y+1+z)]&\\ \to A(y+1+z) \vee [\forall x(x \leqslant (y+1) \to B(x,y+1+z))].&\end{aligned}$$

So $\mathrm{HA} \vdash C(y,z+1) \to C(y+1,z)$. So $\mathrm{HA} \vdash \forall z C(y,z) \to \forall z C(y+1,z)$. By induction, $\mathrm{HA} \vdash \forall z C(y,z)$. Thus $\mathrm{HA} \vdash C(y,0)$. □

Let EE be the Z-open x-ready set generated by all pairs of the form

$$(B(x) \vee \neg x \leqslant z, \forall x(B(x) \vee \neg x \leqslant z)),$$

where $Z = \{z\}$.

Lemma 3.9. *For all* $(B_1,B_2) \in \mathrm{EE}$ *there is a formula B such that*

$$\mathrm{HA} \vdash B_1 \leftrightarrow (x \leqslant z \to B).$$

Proof. We complete the proof following the inductive definition of Z-open x-ready sets. The decidability of $x \leqslant y$ implies that $B(x) \vee \neg x \leqslant z$ is equivalent, modulo HA, to

$x \leqslant z \to B(x)$. Let $(B_1, B_2) \in \mathrm{EE}$ with B_1 (up to HA equivalence) of the form $x \leqslant z \to B$, and let formula A have no free occurrences of x or z. Then $A \to B_1$ is equivalent to $x \leqslant z \to (A \to B)$, and $A \vee B_1$ is equivalent to $A \vee B \vee \neg x \leqslant z$, which is equivalent to $x \leqslant z \to (A \vee B)$. Finally, if y is different from x and z, then $\forall y(x \leqslant z \to B)$ is intuitionistically equivalent to $x \leqslant z \to \forall yB$. □

Note that Lemma 3.9 only needs the decidability of $x \leqslant y$ to work.

Proposition 3.10. HA *satisfies* $\Sigma(\mathrm{EE})$. *So* HA *is strongly complete for its class of end-extension Kripke models.*

Proof. We must prove that $\mathrm{HA} \vdash \forall x B_1 \to B_2$ for all $(B_1, B_2) \in \mathrm{EE}$. We complete the proof following the inductive definition of Z-open x-ready sets. The case for the pairs generating EE obviously holds. Suppose $(B_1, B_2) \in \mathrm{EE}$ is such that $\mathrm{HA} \vdash \forall x B_1 \to B_2$, and A is a formula in which x and z do not occur freely. Then

$$\begin{aligned}\mathrm{HA} \vdash {} & \forall x(A \to B_1) \\ & \to (A \to \forall x B_1) \\ & \to (A \to B_2)\end{aligned}$$

With Lemmas 3.8 and 3.9, there is B such that

$$\begin{aligned}\mathrm{HA} \vdash {} & \forall x(A \vee B_1) \\ & \to \forall x(A \vee (x \leqslant z \to B)) \\ & \to \forall x(x \leqslant z \to (A \vee B)) \\ & \to (A \vee \forall x(x \leqslant z \to B)) \\ & \to (A \vee \forall x B_1) \\ & \to (A \vee B_2)\end{aligned}$$

If y is a variable different from x and z, then

$$\begin{aligned}\mathrm{HA} \vdash {} & \forall x \forall y B_1 \\ & \to \forall y \forall x B_1 \\ & \to \forall y B_2\end{aligned}$$

If the language is countable then, by Theorem 3.7, HA is strongly complete for the class of its end-extension models. The countability restriction can be removed by the methods of Section 4. □

We leave it as a straightforward exercise to show that Proposition 3.10 can be extended to all theories which satisfy the schema of Lemma 3.8 plus the formula $x \leqslant y \vee \neg x \leqslant y$.

Each node k of a Kripke model provides us in a natural way with a classical model $\mathscr{D}_k$ with domain Dk. Equality is interpreted as the congruence implied by the forcing relation of the Kripke model. Following [1], given a classical theory T, a Kripke model is called T-normal if for all its nodes k the model $\mathscr{D}_k$ is a model of T. In case T is Peano arithmetic (PA), we call a PA-normal model *locally PA*.

Proposition 3.11. *There exists a two-node Kripke model $\mathscr{K}$ which is locally* PA *but is not a model of* Σ(EE). *In particular, $\mathscr{K}$ is not a model of* HA.

Proof. Let $\mathscr{M}_\alpha$ be a (classical) countable model, with domain M_α, over the language of arithmetic extended with an extra constant symbol a, satisfying the set of axioms $\mathrm{PA} \cup \{a > n \mid n \in \omega\} \cup \{\neg\,\mathrm{Con}(I\Sigma_a)\} \cup \{\forall x < a\,\mathrm{Con}(I\Sigma_x)\}$. For notations and the existence of such a model, see [1]. With the techniques of [1] one easily shows that for each finite subset $S \subseteq M_\alpha$, there is, over the language of arithmetic extended with extra constant symbols for all elements of M_α and a constant symbol b, a model of $\Delta_0(\mathscr{M}_\alpha) \cup \mathrm{PA} \cup \{b \neq s \mid s \in S\} \cup \{\neg\,\mathrm{Con}(I\Sigma_b)\} \cup \{b < a\} \cup \{\forall x < b\,\mathrm{Con}(I\Sigma_x)\}$. By compactness there is a (classical) countable model $\mathscr{M}_\beta$ with domain M_β, over this same language of arithmetic, satisfying the set of axioms $\Delta_0(\mathscr{M}_\alpha) \cup \mathrm{PA} \cup \{b \neq s \mid s \in M_\alpha\} \cup \{\neg\,\mathrm{Con}(I\Sigma_b)\} \cup \{b < a\} \cup \{\forall x < b\,\mathrm{Con}(I\Sigma_x)\}$. On the set $\{\alpha, \beta\}$, linearly ordered by $\alpha \leqslant \beta$, we construct the obvious Kripke model $\mathscr{K}$ with local structures $\mathscr{M}_\alpha$ and $\mathscr{M}_\beta$. Set A equal to $\mathrm{Pr}_{I\Sigma_{a-1}}(\overline{1=0})$ (intuitionistically stronger than the negation $\neg\,\mathrm{Con}(I\Sigma_{a-1})$ of consistency). Set $B(x)$ equal to $\mathrm{Con}(I\Sigma_x) \to \mathrm{Con}(I\Sigma_{x+1})$. For all $c \in M_\alpha$ we have $\alpha \Vdash \mathrm{Con}(I\Sigma_c)$ if and only if $c < b$ in M_β. So

$$\alpha \nVdash \forall x(x \leqslant (a-2) \to (A \vee B(x))) \to (A \vee \forall x(x \leqslant (a-2) \to B(x))).$$

By Lemma 3.8, $\mathscr{K}$ is not a model of HA. □

$I\Pi_2$-normal Kripke models over a conversely well founded frame are models of $i\Pi_2$; see [10]. In particular, locally PA Kripke models over a finite frame are models of $i\Pi_2$. So Proposition 3.11 implies

Corollary 3.12. *$i\Pi_2$ is not complete with respect to its end-extension Kripke models.*

What about cofinal extension models of HA? By Lemmas 3.2 and 2.4, and by Proposition 2.3, cofinal extension models at least satisfy the schema σ(ce):

$$\forall x(A \vee \forall y(y \leqslant x \to B(y))) \to (A \vee \forall y B(y)),$$

where x is not free in A or $B(y)$.

Proposition 3.13. $\mathrm{HA} \cup \sigma(\mathrm{ce}) \vdash \mathrm{PA}$, *where* PA *is the classical theory of Peano arithmetic.*

Proof. We show that $\mathrm{HA} \cup \sigma(\mathrm{ce}) \vdash A \vee \neg A$, for all formulas A. We complete the proof by induction on the complexity of A. Obviously the case holds over HA for all atomic formulas. The induction steps for the cases $\wedge$, $\vee$, and $\to$ follow from IQC alone since $\mathrm{IQC} \vdash (A \vee \neg A) \wedge (B \vee \neg B) \to ((A \circ B) \vee \neg(A \circ B))$ for all $\circ \in \{\wedge, \vee, \to\}$.

Suppose $\mathrm{HA}\cup\sigma(\mathrm{ce})\vdash\neg A(x)\vee A(x)$, where variable y is not free in $A(x)$. Set $B(x)$ equal to $\exists y\neg A(y)\vee\forall y(y\leqslant x\to A(y))$. By induction, $\mathrm{HA}\cup\sigma(\mathrm{ce})\vdash\neg A(0)\vee A(0)$, so $\mathrm{HA}\cup\sigma(\mathrm{ce})\vdash B(0)$. Similarly, $\mathrm{HA}\cup\sigma(\mathrm{ce})\vdash\neg A(x+1)\vee A(x+1)$ implies $\mathrm{HA}\cup\sigma(\mathrm{ce})\vdash B(x)\to B(x+1)$. By induction $\mathrm{HA}\cup\sigma(\mathrm{ce})\vdash\forall xB(x)$ and thus, by the principle $\sigma(\mathrm{ce})$, $\mathrm{HA}\cup\sigma(\mathrm{ce})\vdash\exists y\neg A(y)\vee\forall yA(y)$. This settles the induction step for the quantifier cases. □

So all cofinal extension models of HA are models of PA.

An analogue to Proposition 3.13 can be found for locally PA Kripke models.

Proposition 3.14. *Every locally* PA *cofinal extension Kripke model is a model of* PA.

Proof. Let $\mathscr{K}$ be a locally PA cofinal extension Kripke model. Let $k\preccurlyeq k'$ be nodes with local classical structures $\mathscr{D}_k$ and $\mathscr{D}_{k'}$ respectively. Then both are (classical) models of PA such that $\mathscr{D}_{k'}$ is a cofinal extension of $\mathscr{D}_k$. By [3, Theorem 7.9], $\mathscr{D}_{k'}$ is an elementary extension of $\mathscr{D}_k$. So $\mathscr{K}$ is a model of classical predicate logic CQC; see for example [9]. So $\mathscr{K}$ is a model of $\mathrm{HA}+\mathrm{CQC}=\mathrm{PA}$. □

Consider a language with binary predicate $x\leqslant y$. A theory which includes both the axiom schema for end-extensions as well as the axiom schema for cofinal extensions, must imply the schema CD for constant domains

$$\forall x(A\vee B(x))\to A\vee\forall xB(x)$$

with x not free in A. Slightly more is true: Let $\sigma(\mathrm{ee})$ be the schema

$$\forall x(x\leqslant y\to(A\vee B(x)))\to(A\vee\forall x(x\leqslant y\to B(x)))$$

for all formulas A and $B(x)$ such that x is not free in A, of Lemma 3.8. Let $\sigma(\mathrm{ce})$ be the schema

$$\forall x(A\vee\forall y(y\leqslant x\to B(y))\to(A\vee\forall yB(y))),$$

where x is not free in A or $B(y)$. We easily verify that

$$\sigma(\mathrm{ee})+\sigma(\mathrm{ce})\vdash\mathrm{CD}.$$

Let PEM_a be the principle of excluded middle for atoms. Then $\mathrm{CD}+\mathrm{PEM}_a\vdash\mathrm{CQC}$, classical predicate logic. So also

$$\sigma(\mathrm{ee})+\sigma(\mathrm{ce})+\mathrm{PEM}_a\vdash\mathrm{CQC}.$$

4. Generalizations to uncountable languages

A special advantage of Kripke models of IQC over many other classes of models of IQC is the option to axiomatize Kripke models in first-order classical logic. This idea, explained below, can already be found in early papers by Grigoriĭ Minc, or Mints; see [5,6].

To each predicate $P(x_1,\ldots,x_n)$ of the language $\mathscr{L}$ of IQC we assign a predicate $P^+(x,x_1,\ldots,x_n)$ of the language $\mathscr{M}$ of classical predicate calculus CQC. The intended meaning of this predicate is $x \Vdash P(x_1,\ldots,x_n)$. To each function symbol $F(x_1,\ldots,x_n)$ of $\mathscr{L}$ we assign a function symbol $F^+(x,x_1,\ldots,x_n)$ of $\mathscr{M}$. The intended meaning of this function symbol is that $x_1,\ldots,x_n$ are elements of domain Dx, and $F^+(x,x_1,\ldots,x_n)\in Dx$. The language $\mathscr{M}$ has a special predicate $D(x,y)$, with intended meaning $y\in Dx$, and a special predicate $x\preccurlyeq x'$ with the obvious meaning. To each formula A of $\mathscr{L}$ in which x is not free, we assign a formula $I(x,A)$ of $\mathscr{M}$ with intended meaning $x\Vdash A$. The formulas $I(x,A)$ are easily defined by induction on the complexity of A. For example, $I(x,P(x_1,\ldots,x_n))$ equals $P^+(x,x_1,\ldots,x_n)$; and $I(x,\forall y B(y))$ equals $\forall x'\forall y(x\preccurlyeq x' \wedge D(x',y)\rightarrow I(x',B(y)))$, with the usual restriction on substitution of variables.

For each set of sentences Σ over $\mathscr{L}$ we define the set $\Sigma^m=\{\forall x I(x,A)\mid A\in\Sigma\}$ over $\mathscr{M}$. For each sentence A over $\mathscr{L}$ define A_m to be the sentence $\exists x\neg I(x,A)$. There is a natural basic set of axioms Λ over $\mathscr{M}$ stating the intended Kripke model axioms. If $\Sigma\nvdash A$ then, by the completeness theorem for Kripke models, $\Sigma^m\cup\{A_m\}\cup\Lambda$ is consistent. The compactness theorem now permits us to remove language cardinality restrictions from certain theorems of the preceding sections.

Theorem 4.1. *Let R be a Z-open x-ready set, and let $\Sigma\cup\{A\}$ be a set of sentences, such that*

- $\Sigma\supseteq\Sigma(R)$; *and*
- $\Sigma\nvdash A$.

Then there exists a Kripke model $\mathscr{K}$ of Σ such that $\mathscr{K}\nvDash A$, and such that $\mathscr{K}$ is an R-bounded extension model.

Proof. Let $\alpha=(B_1(x,\mathbf{y},\mathbf{z}),B_2(\mathbf{y},\mathbf{z}))\in R$, where $\mathbf{z}=(z_j)_j$ lists all free variables from Z that occur in the pair, and $\mathbf{y}=(y_i)_i$ lists all remaining free variables, minus x. Let $B_\alpha\in\mathscr{M}$ be the universal closure of the formula

$$x\preccurlyeq x'\wedge\bigwedge_i D(x',y_i)\wedge\bigwedge_j D(x,z_j)$$
$$\rightarrow[\forall y(D(x,y)\rightarrow I(x',B_1(y,\mathbf{y},\mathbf{z})))\rightarrow I(x',B_2(\mathbf{y},\mathbf{z}))].$$

By Theorem 2.10 each finite subset of $\Sigma^m\cup\{A_m\}\cup\Lambda\cup\{B_\alpha\}_\alpha$ has a model. So by Compactness the whole set has a model $\mathscr{K}$, which is an R-bounded extension model, $\mathscr{K}\models\Sigma$, and $\mathscr{K}\nvDash A$. $\square$

Theorem 2.10 supplies extra properties for the countable case that are not derivable from the theorem as stated above. Although Theorem 4.1 may be strengthened so as to partially capture some of these extra properties, we see no way nice enough to make it worth the effort to do so here.

Theorem 4.2. *Let the language be provided with a binary predicate $x \leqslant y$, and let* CE *be the closed x-ready set of Section* 3.1. *Then the theory* $\Sigma(\mathrm{CE})$ *is sound and strongly complete for the class of cofinal extension Kripke models.*

Proof. Soundness immediately follows from Lemmas 3.2 and 2.4, and Proposition 2.3. Let $\Sigma \cup \{A\}$ be a set of sentences such that $\Sigma \supseteq \Sigma(\mathrm{CE})$, and $\Sigma \nvdash A$. Let B be the universal closure of the formula

$$x \leqslant x' \wedge D(x', y') \rightarrow \exists y(D(x, y) \wedge y' \leqslant y).$$

By Theorem 3.3 each finite subset of $\Sigma^m \cup \{A_m\} \cup \Lambda \cup \{B\}$ has a model. So by Compactness the whole set has a model $\mathscr{K}$, which is a cofinal extension model, $\mathscr{K} \models \Sigma$, and $\mathscr{K} \nvDash A$. $\square$

Theorem 4.3. *Let the language be provided with a predicate* $G(x, \mathbf{z}) = G(x, z_1, \ldots, z_n)$, *where* $x, \mathbf{z}$ *lists all free variables of* G. *Let* GE *be the Z-open x-closed set of Subsection* 3.2. *Then the theory* $\Sigma(\mathrm{GE})$ *is sound and strongly complete for the class of G-expansion Kripke models.*

Proof. Soundness immediately follows from Lemmas 3.6 and 2.4, and Proposition 2.3. let $\Sigma \cup \{A\}$ be a set of sentences such that $\Sigma \supseteq \Sigma(\mathrm{GE})$, and $\Sigma \nvdash A$. Let B be the universal closure of the formula

$$x \leqslant x' \wedge D(x', y') \wedge \bigwedge_j D(x, z_j) \rightarrow (D(x, y') \vee G(y', \mathbf{z})).$$

By Theorem 3.7 each finite subset of $\Sigma^m \cup \{A_m\} \cup \Lambda \cup \{B\}$ has a model. So by Compactness the whole set has a model $\mathscr{K}$, which is a G-expansion model, $\mathscr{K} \models \Sigma$, and $\mathscr{K} \nvDash A$. $\square$

Theorem 4.3 permits us to remove the cardinality restriction in the proof of Proposition 3.10.

The method used in the proofs above to extend a theorem to uncountable languages was also sketched by Dieter Klemke; see [4].

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