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Finding non-trivial elements and splittings in groups. (English summary)
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A finite presentation for a group G is $\langle X \mid R\rangle$ where X is a set of generators and R a set of defining relations. Many interesting algorithmic questions (such as the word problem, the isomorphism problem and the triviality problem) have been shown to be recursively unsolvable for finitely presented groups. However, for some classes of (special) finitely presented groups, such problems are more tractable. For example, the isomorphism problem and the word problem are recursively solvable for finitely presented abelian groups. The motivating question of the paper is the following: Is there a partial algorithm on finite presentations of groups that, on input of a finite presentation P of a nontrivial group, outputs a word w on the generators of P such that w is nontrivial in that group?

The main results of the paper are the following:

Theorem 4.2. Fix any $k>0$. Then there is no algorithm that, on input of a finite presentation $P=\langle X \mid R\rangle$ of a nontrivial group $\overline{P}$, outputs a word w on X of length of at most k such that w is nontrivial in $\overline{P}$. (If P is a group presentation, then $\overline{P}$ denotes the group presented by P.)

Corollary 4.4. Fix any $k>0$. Then there is no algorithm that, on input of a finite presentation $P=\langle X \mid R\rangle$ of a nonabelian group $\overline{P}$, outputs two words w, w' on X, each of length at most k, such that $[w,w']$ is nontrivial in $\overline{P}$.

Theorem 5.5. There is no algorithm that, on input of a finite presentation $P=\langle X \mid R\rangle$ of a group that is a free product of two nontrivial finitely presented groups, outputs two finite presentations P_1, P_2 which present nontrivial groups and whose free product is isomorphic to $\overline{P}$.

The above theorem is used to show that a construction by Stallings on splitting groups with more than one end can never be made algorithmic, nor can the process of decomposing the connect sum of two non-simply connected closed 4-manifolds into non-simply connected summands.

Theorem 6.6. There is no algorithm that, on input of two finite presentations $P=\langle X \mid R\rangle$ and $Q=\langle Y \mid S\rangle$ such that $\overline{P}$ embeds in $\overline{Q}$, outputs an explicit map θ from X to words in Y such that θ extends to an embedding $\overline{\theta}\colon \overline{P} \hookrightarrow \overline{Q}$. *Saeed Salehi*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.