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Unprovability results involving braids. (English summary)
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Some natural and simple facts of mathematics are shown to be unprovable in a certain fragment of (Peano) arithmetic. For $n \geq 2$, the n-strand braid group B_n is the group of isotopy classes of geometric n-strand braids, or the group with the presentation $\langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j|>1$, $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ for $|i-j|=1\rangle$. The authors introduce particular sequences of braids (called $\mathcal{G}_3$-sequences) and show that for each initial braid b in B_3^+, the $\mathcal{G}_3$-sequence from b is finite. But this fact is not provable in $\mathrm{I}\Sigma_1$. They also show an infinite variant of the above fact, and demonstrate its unprovability in $\mathrm{I}\Sigma_2$.

Other unprovability results of the paper concern a combinatorial principle $\mathbf{WO}_f$ for a function f on natural numbers. It is shown that Peano arithmetic proves $\mathbf{WO}_f$ when f is a provably total function of it. Also $\mathbf{WO}_c$ is not provable in $\mathrm{I}\Sigma_1$ when c is a constant function or the squaring function ($c(x) = x^2$). The unprovability of $\mathbf{WO}_f$ in $\mathrm{I}\Sigma_1$ is also proved for some other functions f. *Saeed Salehi*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.