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Growing commas. A study of sequentiality and concatenation. (English summary)

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The paper under review is a contribution to the Theory of Concatenation (TC), which has gotten a new boost since A. Grzegorzczuk's celebrated paper [Studia Logica **79** (2005), no. 2, 163–230; MR2135033 (2005m:03088)] appeared. The author considers a variant of the TC that has a unit element ε with the following axioms:

$$\text{TC}^\varepsilon 1: \varepsilon * x = x \wedge x * \varepsilon = x;$$

$$\text{TC}^\varepsilon 2: (x * y) * z = x * (y * z);$$

$$\text{TC}^\varepsilon 3: x * y = u * v \rightarrow \exists w[(x * w = u \wedge y = w * v) \vee (x = u * w \wedge w * y = v)];$$

$$\text{TC}^\varepsilon 4: a \neq \varepsilon \wedge b \neq \varepsilon \wedge a \neq b;$$

$$\text{TC}^\varepsilon 5: x * y = a \rightarrow (x = \varepsilon \vee y = \varepsilon);$$

$$\text{TC}^\varepsilon 6: x * y = b \rightarrow (x = \varepsilon \vee y = \varepsilon).$$

Then he proves that this theory does not have pairing, meaning that it cannot directly interpret the pairing theory:

$$\text{PAIR1: } \text{pair}(x, y, z) \wedge \text{pair}(x', y', z) \rightarrow (x = x' \wedge y = y');$$

$$\text{PAIR2: } \forall x, y \exists z \text{pair}(x, y, z).$$

A reasonable extension of TC^ε (denoted $\text{TC}^\varepsilon + \text{a-coll}^+$) is sequential, in the sense that it can directly interpret Adjunctive Set Theory:

$$\text{AS1: } \exists x \forall y (y \notin x);$$

$$\text{AS2: } \forall u, v \exists x \forall y [y \in x \leftrightarrow (y \in u \vee y = v)].$$

Thus the theory of concatenation is too weak to be able to code sequences, not even two-element sequences (= pairs), though one can find a natural extension of it which is sequential, capable of coding sequences.

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