Some Fairies in the Incompleteness Wonderland

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The Landscape of Incompleteness

Wuhan, China, 17 August 2021

GÖDEL, ROSSER, KLEENE, CHAITIN, *and* BOOLOS.





KURT GÖDEL.

KURT GÖDEL (1931)
"On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems, I",
Collected Works I (OUP 1986) pp. 135–152.

GÖDEL$_1$ (1931) $Q \vdash G \leftrightarrow \neg\mathtt{Pr}_T(\ulcorner G \urcorner)$, $\mathtt{Pr}_T(x) \equiv$ "x is T-provable".

If $T \vdash G$, then $T \vdash \mathtt{Pr}_T(\ulcorner G \urcorner)$, but also $T \vdash \neg\mathtt{Pr}_T(\ulcorner G \urcorner)$ by G's construction!

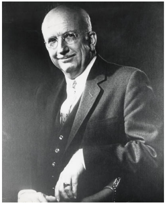
If $\mathbb{N} \nvDash G$, then (by $\mathbb{N} \vDash Q$ we have) $\mathbb{N} \vDash \mathtt{Pr}_T(\ulcorner G \urcorner)$, so $T \vdash G$, contradiction!

GÖDEL$_2$ (1931) $\mathtt{Con}_T \equiv \neg\mathtt{Pr}_T(\ulcorner \bot \urcorner)$.

$T \vdash \mathtt{Pr}_T(\ulcorner G \urcorner) \rightarrow \mathtt{Pr}_T(\ulcorner \neg\mathtt{Pr}_T(\ulcorner G \urcorner) \urcorner)$ by D1,2; also by D3 we have

$T \vdash \mathtt{Pr}_T(\ulcorner G \urcorner) \rightarrow \mathtt{Pr}_T(\ulcorner \mathtt{Pr}_T(\ulcorner G \urcorner) \urcorner)$, thus $T \vdash \mathtt{Con}_T \rightarrow G$, so $T \nvdash \mathtt{Con}_T$.

BARKLEY ROSSER.



BARKLEY ROSSER (1936)
Extensions of Some Theorems of Gödel and Church,
The Journal of Symbolic Logic 1(3):87–91.

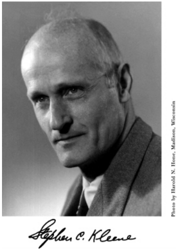
ROSSER (1936) $Q \vdash \mathcal{R} \leftrightarrow \forall x\,[\mathbf{prf}_T(x, \ulcorner\mathcal{R}\urcorner) \rightarrow \exists y < x\,\mathbf{prf}_T(y, \ulcorner\neg\mathcal{R}\urcorner)]$.

$\mathbf{prf}_T(x, u) \equiv$ "x codes a T-proof of u".

If $T \vdash \mathcal{R}$, then $\exists n \mathbb{N} \models \mathbf{prf}_T(n, \ulcorner\mathcal{R}\urcorner)$, and so $T \vdash \exists y < n\,\mathbf{prf}_T(y, \ulcorner\neg\mathcal{R}\urcorner)$!

If $\mathbb{N} \not\models \mathcal{R}$, then $\mathbb{N} \models \exists x\,[\mathbf{prf}_T(x, \ulcorner\mathcal{R}\urcorner) \wedge \cdots]$, and so $T \vdash \mathcal{R}$!

STEPHEN KLEENE.



STEPHEN KLEENE$_1$ (1936)
General Recursive Functions of Natural Numbers,
Mathematische Annalen 112(1):727–742.

STEPHEN KLEENE$_2$ (1950)
A Symmetric Form of Gödel's Theorem,
Indagationes Mathematicae 12:244–246.

KLEENE$_1$ (1936) $K = \text{“}\mathtt{t} \notin \mathcal{W}_{\mathtt{t}}\text{”}$, $\mathcal{W}_{\mathtt{t}} = \{n \in \mathbb{N} \mid T \vdash \text{“}n \notin \mathcal{W}_n\text{”}\}$.

If $T \vdash K$, then $\mathtt{t} \in \mathcal{W}_{\mathtt{t}}$, and so $Q \vdash \text{“}\mathtt{t} \in \mathcal{W}_{\mathtt{t}}\text{”}$ ($\equiv \neg K$), thus $T \vdash \bot$!

If $\mathbb{N} \nvDash K$, then $\mathtt{t} \in \mathcal{W}_{\mathtt{t}}$, and so $T \vdash \text{“}\mathtt{t} \notin \mathcal{W}_{\mathtt{t}}\text{”}$ ($\equiv K$)!

KLEENE$_2$ (1950) $\eta_{(\mathbf{r},\mathbf{s})} = \forall x[\phi_{\mathbf{r}}(\mathbf{r},\mathbf{s})\downarrow_x \rightarrow \exists y < x\, \phi_{\mathbf{s}}(\mathbf{r},\mathbf{s})\downarrow_y]$.

$$\eta_{(u,v)} = \forall x[\phi_u(u,v)\downarrow_x \rightarrow \exists y < x\, \phi_v(u,v)\downarrow_y].$$

$$\phi_{\mathbf{r}}(u,v) = \mu z.\mathtt{prf}_T(z, \ulcorner \eta_{(u,v)} \urcorner), \quad \phi_{\mathbf{s}}(u,v) = \mu z.\mathtt{prf}_T(z, \ulcorner \neg\eta_{(u,v)} \urcorner).$$

GREGORY CHAITIN.

GREGORY CHAITIN (1970)
Computational Complexity and
Gödel's Incompleteness Theorem,
SIGACT News 9(1971):11–12.
Abstract in *Notices AMS* 17:6 (1970) p. 672.

CHAITIN (1970) $\mathscr{K}(w) > e \quad (\forall e \geqslant \mathfrak{C}\, \forall^{\infty} w)$.

$\mathscr{K}(w) = \boldsymbol{\mu} e.[\varphi_e(0) = w]$. $\varphi_{\mathfrak{C}}(x) = \boldsymbol{\pi}_1 \boldsymbol{\mu} y.\mathtt{prf}_T(\boldsymbol{\pi}_2 y, \ulcorner \mathscr{K}(\boldsymbol{\pi}_1 y) > x + \mathfrak{C} \urcorner)$.

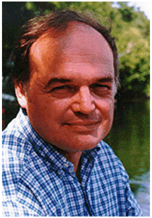
If $T \vdash_p \mathscr{K}(w) > \mathfrak{C}$ with *min* $\langle w, p \rangle$, then $\varphi_{\mathfrak{C}}(0) = w$, and so $\mathscr{K}(w) \leqslant \mathfrak{C}$!

Do Not Use Kleene's Recursion Theorem. $\mathscr{K}(w) = \min\{|\mathcal{P}| : \mathcal{P}\downarrow = w\}$.

$\mathcal{P}_n = \boldsymbol{\pi}_1 \boldsymbol{\mu} y.\mathtt{prf}_T(\boldsymbol{\pi}_2 y, \ulcorner \mathscr{K}(\boldsymbol{\pi}_1 y) > n \urcorner)$. $|\mathcal{P}_n| = \mathfrak{c} + \mathfrak{k} \cdot \ell og_2(n)$. $|\mathcal{P}_N| < N$.

If $T \vdash_p \mathscr{K}(w) > N$ with *min* $\langle w, p \rangle$, then $\mathcal{P}_N\downarrow = w$, and so $\mathscr{K}(w) \leqslant |\mathcal{P}_N| < N$!

GEORGE BOOLOS.



GEORGE BOOLOS (1989)
A New Proof of the Gödel Incompleteness Theorem,
Notices AMS 36(4):388–390.

BOOLOS (1989) $\boldsymbol{\beta}^{<10\cdot\ell}(\mathfrak{b})$.

$\mathrm{Def}(n,\varphi) \equiv T \vdash \forall\xi[\varphi(\xi) \leftrightarrow \xi = n]$. $\mathcal{D}^{<y}(x) \equiv \exists|\varphi| < y\,\mathrm{Def}(x,\varphi)$.
$\boldsymbol{\beta}^{<y}(x) \equiv [x = \boldsymbol{\mu} z.\neg\mathcal{D}^{<y}(z)]$. $\ell = |\boldsymbol{\beta}^{<y}(x)|$. $|\boldsymbol{\beta}^{<10\cdot\ell}(x)| < 10\cdot\ell$.
If $T \vdash \boldsymbol{\beta}^{<10\cdot\ell}(\mathfrak{b})$, then $T \vdash \forall\xi[\boldsymbol{\beta}^{<10\cdot\ell}(\xi) \leftrightarrow \xi = \mathfrak{b}]$, so $\mathcal{D}^{<10\cdot\ell}(\mathfrak{b})$!

Π_1-INCOMPLETENESS.

Fix a sufficiently strong base theory $\mathfrak{B}$[1] (on a sufficiently expressive language).

- All theories (T, U, $\cdots$) will be RE extensions of $\mathfrak{B}$.
- RE theories are Σ_1-definable; and so Δ_0-definable (CRAIG).

Def. For a Δ_0-formula $\boldsymbol{\tau}(x)$ let $\mathfrak{Th}_{\boldsymbol{\tau}} = \{\theta \mid \mathbb{N} \vDash \boldsymbol{\tau}(\ulcorner\theta\urcorner)\}$.

- We consider Δ_0-formulae $\boldsymbol{\tau}(x)$ such that $\mathfrak{B} \subseteq \mathfrak{Th}_{\boldsymbol{\tau}} \nvdash \bot$.

Definition

A Π_1-incompleteness is a mapping $\boldsymbol{\tau} \mapsto \theta_{\boldsymbol{\tau}}$ which assigns a Π_1-sentence $\theta_{\boldsymbol{\tau}}$ to a Δ_0-formula $\boldsymbol{\tau}(x)$ such that if $\mathfrak{B} \subseteq \mathfrak{Th}_{\boldsymbol{\tau}} \nvdash \bot$, then $\theta_{\boldsymbol{\tau}}$ is true and $\mathfrak{Th}_{\boldsymbol{\tau}}$-unprovable, i.e., (*i*) $\mathbb{N} \vDash \theta_{\boldsymbol{\tau}}$ and (*ii*) $\mathfrak{Th}_{\boldsymbol{\tau}} \nvdash \theta_{\boldsymbol{\tau}}$. ▲

Remark

If $\mathfrak{Th}_{\boldsymbol{\tau}}$ is (also) Σ_1-sound ($\equiv$ 1-consistent, or is ω-consistent), then we also have (*iii*) $\mathfrak{Th}_{\boldsymbol{\tau}} \nvdash \neg\theta_{\boldsymbol{\tau}}$. ▲

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[1] which could be Peano's Arithmetic, or $I\Sigma_1$, or $I\Delta_1(\equiv EA+B\Sigma_1)$.

Some Instances of Π_1-INCOMPLETENESS Witnesses.

GÖDEL$_1$ (1931) $\tau \mapsto \mathbb{G}_\tau$

GÖDEL$_2$ (1931) $\tau \mapsto \texttt{Con}_\tau$

ROSSER (1936) $\tau \mapsto \mathbb{R}_\tau$

KLEENE$_1$ (1936) $\tau \mapsto \mathbb{K}^1_\tau$

KLEENE$_2$ (1950) $\tau \mapsto \mathbb{K}^2_\tau$

CHAITIN (1970) $\tau \mapsto \mathbb{C}_\tau$

BOOLOS (1989) $\tau \mapsto \mathbb{B}_\tau$

- $\tau \mapsto \texttt{Con}(\tau + \neg\texttt{Con}_\tau)$

$\equiv \texttt{Con}_\tau$

Properties of Π_1-INCOMPLETENESS WITNESSES.

Definition

A Π_1-incompleteness witnesses is said (to be)

constructive if $\tau \mapsto \theta_\tau$ is a constructive (effective / recursive) mapping.

ROSSER*ian* if $\mathfrak{Th}_\tau \nvdash \neg\theta_\tau$ when $\mathfrak{B} \subseteq \mathfrak{Th}_\tau \nvdash \bot$ (no need for $1/\omega$-Con).

$\Rightarrow$ GÖDEL$_2$ if $\mathfrak{Th}_\tau \vdash \texttt{Con}_\tau \rightarrow \theta_\tau$ (so, $\mathfrak{Th}_\tau \nvdash \theta_\tau \Rightarrow \mathfrak{Th}_\tau \nvdash \texttt{Con}_\tau$).

GÖDEL$_2\Rightarrow$ if $\mathfrak{Th}_\tau \vdash \theta_\tau \rightarrow \texttt{Con}_\tau$ (so, $\mathfrak{Th}_\tau \nvdash \texttt{Con}_\tau \Rightarrow \mathfrak{Th}_\tau \nvdash \theta_\tau$).

[$\approxeq$ GÖDEL$_2$ if $\mathfrak{Th}_\tau \vdash \texttt{Con}_\tau \leftrightarrow \theta_\tau$ (so, $\mathfrak{Th}_\tau \nvdash \theta_\tau \Leftrightarrow \mathfrak{Th}_\tau \nvdash \texttt{Con}_\tau$)].

formalizable if $\mathfrak{Th}_\tau + \texttt{Con}_\tau \vdash \theta_\tau \wedge \neg\texttt{Pr}_\tau(\ulcorner\theta_\tau\urcorner)$ when $\mathfrak{Th}_\tau \nvdash \neg\texttt{Con}_\tau$.

Three Short Stories around Gödel's Incompleteness Theorems (in Japanese) by M. KIKUCHI & T. KURAHASHI (2011). ▲

On Constructivity and the Rosser Property.

- S. Salehi & P. Seraji, On Constructivity and the Rosser Property: a closer look at some Gödelean proofs, *Annals of Pure and Applied Logic* 169:10 (2018) 971–980.

incompleteness	constructivity	Rosser property
Gödel$_1$ (1931)	+	−
Gödel$_2$ (1931)	+	−
Kleene$_1$ (1936)	+	−
Rosser (1936)	+	+
Kleene$_2$ (1950)	+	+
Chaitin (1970)	−	+
Boolos (1989)	−	−

How $\mathbb{G}$ÖDEL$_2$ is Derived and What $\mathbb{G}$ÖDEL$_2$ Delivers.

- S. SALEHI, Gödel's Second Incompleteness Theorem: how it is derived and what it delivers, *The Bulletin of Symbolic Logic* 26:3-4 (2020) 241–256.

incompleteness	it derives $\mathbb{G}_2$	$\mathbb{G}_2$ delivers it
$\mathbb{G}$ÖDEL$_1$ (1931)	+	+
$\mathbb{K}$LEENE$_1$ (1936)	+	+
$\mathbb{R}$OSSER (1936)	+	−
$\mathbb{K}$LEENE$_2$ (1950)	+	−
$\mathbb{C}$HAITIN (1970)	−	−
$\mathbb{B}$OOLOS (1989)	−	+

Comparing Π_1-Incompleteness Witnesses with each other.

Definition

Let $\Theta\colon \boldsymbol{\tau} \mapsto \theta_{\boldsymbol{\tau}}$ and $\Psi\colon \boldsymbol{\tau} \mapsto \psi_{\boldsymbol{\tau}}$ be two Π_1-incompleteness witnesses. We say that Θ *is derived from* Ψ, or Ψ *delivers* Θ, denoted $\Theta \preccurlyeq \Psi$, when for every system $\boldsymbol{\tau}$ we have $\mathfrak{Th}_{\boldsymbol{\tau}} \vdash \theta_{\boldsymbol{\tau}} \rightarrow \psi_{\boldsymbol{\tau}}$.

$$\text{So, } \mathfrak{Th}_{\boldsymbol{\tau}} \nvdash \psi_{\boldsymbol{\tau}} \Longrightarrow \mathfrak{Th}_{\boldsymbol{\tau}} \nvdash \theta_{\boldsymbol{\tau}}.$$

Let $\Theta \approxeq \Psi$ abbreviate $\Theta \preccurlyeq \Psi \preccurlyeq \Theta$, and $\Theta \precnapprox \Psi$ shorten $\Theta \preccurlyeq \Psi \not\preccurlyeq \Theta$. ▲

Theorem

$$\mathbb{C} \succnapprox \widetilde{\mathbb{B}} \precnapprox \mathbb{G}_{\mathbf{2}} \approxeq \mathbb{K}^1 \approxeq \mathbb{G} \precnapprox \overline{\mathbb{R}} \approxeq \overline{\mathbb{K}^2}.$$ ■

$\overline{\mathbb{R}}$, $\overline{\mathbb{K}^2}$ alternative. $\widetilde{\mathbb{B}}$ substantial variant.

Boolos is the weakest, derivable from all.
Rosser is almost the strongest, delivers all except Chaitin.
Chaitin is the most neutral, not derived from any, and delivers no other except Boolos.

Some Instances of Π_2-INCOMPLETENESS Witnesses.

- H. PUTNAM (2000); Nonstandard Models and Kripke's Proof of the Gödel Theorem, *Notre Dame J. Formal Logic* 41(1):53–58.

  Received June 5, 2001; printed July 15, 2002.

  "**Acknowledgments**

  This paper is developed from a lecture to the Department of Computer Science at Peking University, June 1984. I have decided to publish this lecture at this time because Kripke's proof [date?] is *still* unpublished."

- H. KOTLARSKI (1998); Other Proofs of Old Results, *Mathematical Logic Quarterly* 44(4):474–480.

  For every RE theory there exists a function that dominates all the provably total functions of that theory.

Thank You!

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