Gödel's Incompleteness Phenomenon; Computationally

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Gödel's Incompleteness Phenomenon; Computationally 14th CLMPS'2011 (1/22)

Completeness vs. Incompleteness of Kurt Gödel

- **Completeness** of Logic $\mathfrak{L}$ w.r.t Class of Structures $\mathscr{K}$:

For any formula φ: $\forall \mathcal{M} \in \mathscr{K} (\mathcal{M} \models \varphi) \implies \vdash_{\mathfrak{L}} \varphi$.

- **Strong Completeness**

For any theory Γ (set of formulas) and any formula φ:

$$\forall \mathcal{M} \in \mathscr{K} (\mathcal{M} \models \Gamma \Rightarrow \mathcal{M} \models \varphi) \implies \Gamma \vdash_{\mathfrak{L}} \varphi.$$

- **Soundness** of Logic $\mathfrak{L}$ w.r.t Class of Structures $\mathscr{K}$:

For any formula φ: $\vdash_{\mathfrak{L}} \varphi \implies \forall \mathcal{M} \in \mathscr{K} (\mathcal{M} \models \varphi)$.

- $\equiv$ Strong Soundness
- $\forall \Gamma \, \forall \varphi :$ $\Gamma \vdash_{\mathfrak{L}} \varphi \Longrightarrow \forall \mathcal{M} \in \mathscr{K} (\mathcal{M} \models \Gamma \Rightarrow \mathcal{M} \models \varphi)$.

So (*here*) Completeness & Soundness are Semantic concepts.

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Completeness vs. Incompleteness of Kurt Gödel

- **Completeness** of Theory T w.r.t Class of Structures $\mathscr{K}$:

For any formula φ: $\forall \mathcal{M} \in \mathscr{K} (\mathcal{M} \models \mathrm{T} \Rightarrow \mathcal{M} \models \varphi) \Longrightarrow \mathrm{T} \vdash_{\mathfrak{L}} \varphi$.

- **Soundness** of Theory T w.r.t Class of Structures $\mathscr{K}$:

For any formula φ: $\mathrm{T} \vdash_{\mathfrak{L}} \varphi \Longrightarrow \forall \mathcal{M} \in \mathscr{K} (\mathcal{M} \models \mathrm{T} \Rightarrow \mathcal{M} \models \varphi)$.

The Theory T axiomatizes the Class $\mathscr{K}$:
T is Sound and Complete w.r.t $\mathscr{K}$; $\mathrm{T} = \mathtt{AxTh}(\mathscr{K})$; $\mathscr{K} = \mathtt{Mod}(\mathrm{T})$.

(SEMANTIC) $\mathscr{K}$ is axiomatizable iff $\mathscr{K} = \mathtt{Mod}(\mathtt{Th}(\mathscr{K}))$ iff $\mathscr{K}$ is closed under elementary equivalence and ultra-products iff $\mathscr{K}$ is an elementary class.

(SYNTACTIC) $\mathtt{Der}(\mathrm{T}) = \{\theta \mid \mathrm{T} \vdash \theta\} = \mathtt{Th}(\mathtt{Mod}(\mathrm{T}))$.

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- Syntactic Completeness of Theory T:

For any formula φ: either $\mathrm{T} \vdash_{\mathfrak{L}} \varphi$ or $\mathrm{T} \vdash_{\mathfrak{L}} \neg\varphi$.

That is *Negation* Completeness: $\mathrm{T} \vdash_{\mathfrak{L}} \neg\varphi \Longleftarrow \mathrm{T} \nvdash_{\mathfrak{L}} \varphi$.
Conjunction Completeness: $\mathrm{T} \vdash_{\mathfrak{L}} \varphi \wedge \psi \iff \mathrm{T} \vdash_{\mathfrak{L}} \varphi \,\&\, \mathrm{T} \vdash_{\mathfrak{L}} \psi$.
Disjunction Completeness: $\mathrm{T} \vdash_{\mathfrak{L}} \varphi \vee \psi \iff \mathrm{T} \vdash_{\mathfrak{L}} \varphi \text{ or } \mathrm{T} \vdash_{\mathfrak{L}} \psi$.
Implication Completeness: $\mathrm{T} \vdash_{\mathfrak{L}} \varphi \rightarrow \psi \iff \mathrm{T} \nvdash_{\mathfrak{L}} \varphi \text{ or } \mathrm{T} \vdash_{\mathfrak{L}} \psi$.
Universal Completeness: $\mathrm{T} \vdash_{\mathfrak{L}} \forall \mathrm{x}\varphi \iff \mathrm{T} \vdash_{\mathfrak{L}} \varphi(\mathrm{x}) \text{ for all } x$.
Existential Completeness: $\mathrm{T} \vdash_{\mathfrak{L}} \exists \mathrm{x}\varphi \iff \mathrm{T} \vdash_{\mathfrak{L}} \varphi(\mathrm{t}) \text{ for an } t$.

It all makes sense in the case of

- Consistency of Theory T:

For any formula φ: either $\mathrm{T} \nvdash_{\mathfrak{L}} \varphi$ or $\mathrm{T} \nvdash_{\mathfrak{L}} \neg\varphi$.

$$\mathrm{T} \vdash_{\mathfrak{L}} \neg\varphi \Longrightarrow \mathrm{T} \nvdash_{\mathfrak{L}} \varphi.$$

Completeness vs. Incompleteness of Kurt Gödel

(SYNTACTIC) Completeness and Consistency $\equiv$
(SEMANTIC) Completeness and Soundness w.r.t a Class of Equivalent Models.

$\equiv \forall\varphi: \quad T \vdash \neg\varphi \iff T \nvdash \varphi$.

- (Syn.) Complete + Consistent $\iff$ Maximally Consistent.
- So, by Axiom of Choice, every Theory *can be* COMPLETED.
  `But not in an effective (algorithmic) way !`

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Completeness vs. Incompleteness of Kurt Gödel

- Axiomatizable Theory: A Consistent Theory whose Axioms can be Algorithmically Listed (be Recursively Enumerable).
  - Then, the Theorems of the Theory will be R.E. too.
- A(n Axiomatizable) Theory is called *Decidable* if the set of its Theorems is Decidable (Recursive).
- A(n Axiomatizable) Theory T is *Completable* if there exists a(n axiomatizable) Complete Theory T' extending $\mathrm{T}(\subseteq \mathrm{T}')$.

From a Logician's Point of View:

- (SYNTACTIC) Complete $\Longrightarrow$ Decidable $\Longrightarrow$ Completable.

Completeness vs. Incompleteness of Kurt Gödel

- T is Complete $\Longrightarrow$ T is Decidable:

Since $\{\theta \mid \mathrm{T} \vdash \theta\}$ is R.E. then $\{\theta \mid \mathrm{T} \nvdash \theta\} = \{\theta \mid \mathrm{T} \vdash \neg\theta\}$ is R.E.
So, $\{\theta \mid \mathrm{T} \vdash \theta\}$ is Decidable (Recursive).

- T is Decidable $\Longrightarrow$ T is Completable:

The Henkin Construction for a Completion of T is `effective`, when T is Decidable.

Thus that Completion is also Decidable; so T is Completable.

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Completeness vs. Incompleteness of Kurt Gödel

- Does Decidability (of T) $\Longrightarrow$ Completeness (of T)?

NO: Monadic Predicate Logic (without Equality –
Unary Relations Only [like $P(x)$]).
Decidable but Incomplete ($\nvdash \forall x P(x)$ & $\nvdash \exists x \neg P(x)$).

- Completeness $\Longrightarrow$ Decidability.

$\not\Longleftarrow$

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Completeness vs. Incompleteness of Kurt Gödel

- Does Completability (of T) $\Longrightarrow$ Decidability (of T)?

NO: First-Order Logic with equality is UNDecidable

(by Church's Theorem)

but Completable:

$$\text{Logic} + \forall x \forall y (x = y).$$

- Decidability $\Longrightarrow$ Completability.

$$\not\Longleftarrow$$

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Completeness vs. Incompleteness of Kurt Gödel

- Incompletability $\Longrightarrow$ Undecidability $\Longrightarrow$ Incompleteness
  $\not\Longleftarrow$ $\not\Longleftarrow$

▶ Incompletable = Essentially Undecidable

A Simple Example of an Incompletable Theory ?
With a Simple Proof of its Incompletability?
●●● Gödel's Incompleteness Theorem ...

Kurt Gödel's Incompleteness . . . COMPUTATIONALLY

A Complete Theory

Axioms A_L over the language $\langle 0, \mathbf{S}, < \rangle$:

- $\forall x \forall y (x < y \rightarrow y \not< x)$
- $\forall x \forall y \forall z (x < y \wedge y < z \rightarrow x < z)$
- $\forall x \forall y (x < y \vee x = y \vee y < x)$
- $\forall x (x \not< 0)$
- $\forall x \forall y (x < \mathbf{S}(y) \leftrightarrow x < y \vee x = y)$
- $\forall x (x \neq 0 \rightarrow \exists y [y = \mathbf{S}(x)])$

This Axiomatizes the Theory $\langle \mathbb{N}, 0, \mathbf{S}, < \rangle$.

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A Ternary Predicate $\mathcal{T}(e,x,t)$=
The (single-input) Algorithm (with code) e with input x takes time t to halt (and it indeed halts).

Let the Theory A_S be A_L +

$$\{\mathcal{T}(\overline{e},\overline{x},\overline{t}) \mid \mathbb{N} \models \mathcal{T}(e,x,t)\}$$

where $\overline{n}$ is $\underbrace{\mathbf{S}\cdots\mathbf{S}}_{n \text{ times}}(0)$.

Theory A_S is UnDecidable but Completable.

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A Completion:

$$A_S + \forall y \forall x \forall z \mathcal{T}(y, x, z).$$

UnDecidability of A_S:
Was A_S decidable then Halting Problem would be solvable:
Take e and x, form $\varphi_{e,x} = \exists t \mathcal{T}(\overline{e}, \overline{x}, z)$.
$A_S \vdash \varphi_{e,x} \iff \mathbb{N} \models \mathcal{T}(e, x, t)$ for some $t \in \mathbb{N} \iff$
Program e with Input x eventually halts.

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UnDecidability of A_S Directly:

If $\{\theta \mid A_S \vdash \theta\}$ is Decidable, then so is

$$\mathfrak{D} = \{n \mid A_S \nvdash \exists z \mathcal{T}(\overline{n}, \overline{n}, z)\}.$$

Let the Algorithm (with code) e halt on x whenever $x \in \mathfrak{D}$ and does not halt (loop forever) whenever $x \notin \mathfrak{D}$.

Then Algorithm (with code) e with input e:

- (Algorithm e Halts in time t on input e) $\iff$
  $\iff [\mathbb{N} \models \mathcal{T}(n, n, t)] \iff [\mathcal{T}(\overline{n}, \overline{n}, \overline{t}) \in A_S] \iff$
  $\iff [A_S \vdash \exists z \mathcal{T}(\overline{e}, \overline{e}, z)] \iff [e \notin \mathfrak{D}] \iff$
  (Algorithm e does NOT halt on input e)!

The Proof Works for Every Sound $T \supseteq A_S$ (s.t. $\mathbb{N} \models T$).

So, A_S is NOT Soundly Completable.

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So, we can complete A_S as $A_S + \forall y \forall x \forall z \mathcal{T}(y, x, z)$.
But there is no complete $T \supseteq A_S$ such that $\mathbb{N} \models T$.
Thus $\mathtt{Th}(\mathbb{N}, 0, \mathbf{S}, <, \mathcal{T})$ is NOT R.E.

The Proof is the Classical Argument:
A Sound Theory (of $\mathbb{N}$) Can Not Be Complete: Because of the Existence of a Definable non-E.R. Set, or an R.E. Set Which is Not Decidable. For example, $K = \{n \in \mathbb{N} \mid n \in W_n\}$ is R.E. and UnDecidable. Thus $\overline{K} = \{n \mid n \notin W_n\}$ is not R.E. For a Sound Theory T, we have the R.E. Set $\{m \mid T \vdash \text{``}m \notin W_m\text{''}\} \subset \overline{K}$. So, there must Exist some $n \in \overline{K}$ for which $T \nvdash \text{``}n \notin W_n\text{''}$. Thus $(\mathbb{N} \models)\text{``}n \notin W_n\text{''}$ is a True Sentence which is Not $T-$Provable.

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Kurt Gödel's Incompleteness . . . COMPUTATIONALLY

Let A_T be A_S + $\{\neg\mathcal{T}(\overline{e},\overline{x},\overline{t}) \mid \mathbb{N} \models \neg\mathcal{T}(e,x,t)\}$
in a Language that Contains a (Definable) Pairing Function π.

So, A_T is Axiomatized over $\langle 0, \mathbf{S}, <, \mathcal{T}, \pi\rangle$ by

- $\forall x \forall y (x < y \rightarrow y \not< x)$
- $\forall x \forall y \forall z (x < y \wedge y < z \rightarrow x < z)$
- $\forall x \forall y (x < y \vee x = y \vee y < x)$
- $\forall x (x \not< 0)$
- $\forall x \forall y (x < \mathbf{S}(y) \leftrightarrow x < y \vee x = y)$
- $\forall x (x \neq 0 \rightarrow \exists y [y = \mathbf{S}(x)])$
- $\{\ \mathcal{T}(\overline{e},\overline{x},\overline{t}) \mid \mathbb{N} \models \mathcal{T}(e,x,t)\}$
- $\{\neg\mathcal{T}(\overline{e},\overline{x},\overline{t}) \mid \mathbb{N} \models \neg\mathcal{T}(e,x,t)\}$
- $\forall x \forall y \forall u \forall v \Big(\pi(x,y) = \pi(u,v) \iff x = u \wedge y = v\Big)$

Kurt Gödel's Incompleteness . . . COMPUTATIONALLY

Theory A_T is Consistent (and $\mathbb{N}-$Sound) but INCOMPLETABLE:

- Let $\varphi_{\langle k,l\rangle} = \exists x[\mathcal{T}(\overline{k},\pi(\overline{k},\overline{l}),x) \wedge \forall y \leqslant x \neg\mathcal{T}(\overline{l},\pi(\overline{k},\overline{l}),y)]$.
- If $\mathrm{T} \supseteq A_T$ is Complete (Not-Sound), then are Decidable: $\{\langle k,l\rangle \mid \mathrm{T} \vdash \varphi_{\langle k,l\rangle}\}$ and $\{\langle k,l\rangle \mid \mathrm{T} \vdash \neg\varphi_{\langle k,l\rangle}\}$.

- Let Algorithm (with code) m on input $\langle k,l\rangle$ Halt, Whenever $\mathrm{T} \vdash \varphi_{\langle k,l\rangle}$ and Never Halt Whenever $\mathrm{T} \nvdash \varphi_{\langle k,l\rangle}$.
- Let Algorithm (with code) n on input $\langle k,l\rangle$ Halt, Whenever $\mathrm{T} \vdash \neg\varphi_{\langle k,l\rangle}$ and Never Halt Whenever $\mathrm{T} \nvdash \neg\varphi_{\langle k,l\rangle}$.

Kurt Gödel's Incompleteness . . . COMPUTATIONALLY

Algorithm (with code) m on input $\langle k, l\rangle$ Halts Whenever $\mathrm{T} \vdash \varphi_{\langle k,l\rangle}$ and Never Halts Whenever $\mathrm{T} \nvdash \varphi_{\langle k,l\rangle}$.
Algorithm (with code) n on input $\langle k, l\rangle$ Halts Whenever $\mathrm{T} \vdash \neg\varphi_{\langle k,l\rangle}$ and Never Halts Whenever $\mathrm{T} \nvdash \neg\varphi_{\langle k,l\rangle}$.

Consider $\varphi_{\langle n,m\rangle}$: Was T Complete, then
either $\mathrm{T} \vdash \varphi_{\langle n,m\rangle}$ or $\mathrm{T} \vdash \neg\varphi_{\langle n,m\rangle}$.

We Will Get A Contradiction For Each Case ...

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Kurt Gödel's Incompleteness . . . COMPUTATIONALLY

- If $\mathrm{T} \vdash \varphi_{\langle n,m\rangle}$ Then $\mathrm{T} \nvdash \neg\varphi_{\langle n,m\rangle}$. Thus $\mathcal{T}(m,\pi(n,m),t)$ holds for some t and $\neg\mathcal{T}(n,\pi(n,m),s)$ holds for every s. Also $\mathrm{T} \vdash \exists x[\mathcal{T}(\overline{n},\pi(\overline{n},\overline{m}),x) \wedge \forall y \leqslant x \neg\mathcal{T}(\overline{m},\pi(\overline{n},\overline{m}),y)]$. Since $\mathrm{T} \vdash \mathcal{T}(\overline{m},\pi(\overline{n},\overline{m}),\overline{t})$, then $x_0 < \overline{t}$. Whence, $\bigvee_{\{i<\overline{t}\}} x_0 = \overline{i}$, but then $A_T \vdash \bigwedge_{\{i<t\}} \neg\mathcal{T}(\overline{n},\pi(\overline{n},\overline{m}),\overline{i})$, so $\mathrm{T} \vdash \neg\mathcal{T}(\overline{n},\pi(\overline{n},\overline{m}),x_0)$. Contradiction!
- If $\mathrm{T} \vdash \neg\varphi_{\langle n,m\rangle}$ Then $\mathrm{T} \nvdash \varphi_{\langle n,m\rangle}$. Thus $\mathcal{T}(n,\pi(n,m),t)$ holds for some t and $\neg\mathcal{T}(m,\pi(n,m),s)$ holds for every s. Also $\mathrm{T} \vdash \forall x[\mathcal{T}(\overline{n},\pi(\overline{n},\overline{m}),x) \rightarrow \exists y \leqslant x \mathcal{T}(\overline{m},\pi(\overline{n},\overline{m}),y)]$. Since $A_T \vdash \mathcal{T}(\overline{n},\pi(\overline{n},\overline{m}),\overline{t})$, then $\mathrm{T} \vdash \mathcal{T}(\overline{m},\pi(\overline{n},\overline{m}),y_0)$ for some $y_0 \leqslant \overline{t}$. But then $\bigvee_{\{i\leqslant t\}} y_0 = \overline{i}$ and $A_T \subseteq \mathrm{T} \vdash \bigwedge_{\{i\leqslant t\}} \neg\mathcal{T}(\overline{m},\pi(\overline{n},\overline{m}),\overline{i})$. Contradiction!

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Kurt Gödel's Incompleteness . . . COMPUTATIONALLY

- ▷ The Proof Resembles Rosser's Strengthening of
  Gödel's Theorem for ALL CONSISTENT Theories,
  instead of Sound or $\omega-$CONSISTENT Theories.
- ▷ The Proof is Effective:
  For any (Hypothetical Code for) Enumeration of T, one can
  effectively find a (Gödel-Rosser) $T-$independent Sentence.
- ▷ Any Theory Capable of Interpreting A_T is INCOMPLETABLE
  = Essentially Undecidable=Essentially Incomplete.
  Like Robinson's Arithmetic *Q* or *PRA* or ...
- ▷ In the Proof Was Avoided:
  Coding of Syntax (Coding of Algorithms Was Needed)
  Constructing Gödel Sentence (`I Am Not Provable`)
  Finding a Fixed Point Formula (`Diagonalization`)

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Kurt Gödel's Incompleteness . . . COMPUTATIONALLY

- ▷ By Relativizing the arguments to a DEFINABLE ORACLE
  Tarski's Theorem on the Undefinability of Truth.
- ▷ By Finitely Axiomatizing A_T
  Church's Theorem on the Undecidability of Logic.
- ▷ By Finitely Axiomatizing A_T
  We Find a Theory A_U such that
  Gödel's Second Incompleteness Theorem
  Can Be Proved For Every Theory $T \supseteq A_U$.
- ▷ Then One Can Also Prove Rice's Theorem
  For R.E. Theories ...

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Gödel's Incompleteness Phenomenon; Computationally 14th CLMPS'2011 (21/22)

Thank You!

Thanks to

The Participants .... For Listening...

and

The Organizers ... For Taking Care of Everything...

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