# The Fundamental Theorem of Algebra — Logically

Saeed Salehi

University of Tabriz and IPM

http://SaeedSalehi.ir/

30 October 2014

Institute for Research in Fundamental Sciences

## The Theorem

Every (non-trivial) Polynomial Has a Complex Root.

Coefficients can be real or complex.

## Logically

a first-order scheme

$$\forall \mathbf{a_1}, \mathbf{a_2}, \cdots, \mathbf{a_{n-1}}, \mathbf{a_n} \exists x \big( x^n + \mathbf{a_n} x^{n-1} + \mathbf{a_{n-1}} x^{n-2} + \cdots + \mathbf{a_2} x + \mathbf{a_1} = 0 \big)$$

$$\mathbf{n} = 1, 2, 3, \cdots$$

## More Logically

1. $\forall a_1 \exists x (x + a_1 = 0)$
2. $\forall a_1, a_2 \exists x (x^2 + a_2 x + a_1 = 0)$
3. $\forall a_1, a_2, a_3 \exists x (x^3 + a_3 x^2 + a_2 x + a_1 = 0)$
4. $\forall a_1, a_2, a_3, a_4 \exists x (x^4 + a_4 x^3 + a_3 x^2 + a_2 x + + a_1 = 0)$
5. $\forall a_1, a_2, a_3, a_4, a_5 \exists x (x^5 + a_5 x^4 + a_4 x^3 + a_3 x^2 + a_2 x + a_1 = 0)$

$\vdots$

(n) $\forall \bar{a} \exists x (x^n + \sum_{i=1}^{i=n} a_i x^{i-1} = 0)$

$\vdots$

## Axiom / Axiomatic / Axiomaitzation

Merriam-Webster: www.merriam-webster.com

AXIOM:

a statement accepted as true as the basis for argument or inference

*Postulate*

AXIOMATIC:

based on or involving an axiom or system of axioms

AXIOMATIZATION:

the act or process of reducing to a system of axioms

## Axiom / Axiomatic / Axiomaitze

Oxford: www.oxforddictionaries.com

AXIOM:
a statement or proposition which is regarded as being established, accepted, or self-evidently true *the axiom that sport builds character*
Math: a statement or proposition on which an abstractly defined structure is based
Origin: late 15th century: from French axiome or Latin axioma, from Greek axio-ma 'what is thought fitting', from axios 'worthy'

AXIOMATIC: self-evident or unquestionable
*it is axiomatic that good athletes have a strong mental attitude*
Math: relating to or containing axioms

AXIOMATIZE: express (a theory) as a set of axioms
*the attempts that are made to axiomatize linguistics*

## Axiomatizing Mathematical Structures

### Addition and Multiplication of the Complex Numbers $\langle\mathbb{C},+,\cdot\rangle$

Tarski: The (First-Order Logical) Theory of the Structure $\langle\mathbb{C},0,1,-,^{-1},+,\cdot\rangle$ is Decidable and Can Be Axiomatized As an **Algebraically Closed Field** with zero characteristic.

- $x+(y+z)=(x+y)+z$
- $x+y=y+x$
- $x+0=x$
- $x+(-x)=0$
- $x\cdot(y+z)=(x\cdot y)+(x\cdot z)$
- $x\cdot(y\cdot z)=(x\cdot y)\cdot z$
- $x\cdot y=y\cdot x$
- $x\cdot 1=x$
- $x\neq 0\rightarrow x\cdot x^{-1}=1$
- $0\neq 1+\cdots+1=n$

- $\forall a_1,\cdots,a_n\exists x(x^n+\sum_{i=1}^{i=n}a_ix^{i-1}=0)$ $\qquad n=1,2,\cdots$

## Some References


- G. KREISEL, J. L. KRIVINE, *Elements of Mathematical Logic: model theory*, North Holland 1967.
- Z. ADAMOWICZ, P. ZBIERSKI, *Logic of Mathematics: a modern course of classical logic*, Wiley 1997.
- J. BOCHNAK, M. COSTE, M.-F. ROY, *Real Algebraic Geometry*, Springer 1998.
- S. BASU, R. POLLACK, M.-F. COSTE-ROY, *Algorithms in Real Algebraic Geometry*, 2nd ed. Springer 2006.

## Algebraic Geometry

$\mathbb{R}$ and $\mathbb{C}$ with $+$ and $\cdot$

Tarski & Chevalley:

The projection of a constructible set (in $\mathbb{C}$) is constructible.

Constructible:

Boolean ($^\complement$, $\cap$, $\cup$) Combinations of $\{\overline{x} \mid \mathsf{p}(\overline{x}) = 0\}$'s.

Tarski & Seidenberg:

The projection of a semi-algebraic set (in $\mathbb{R}$) is semialgebraic.

Semi-Algebraic:

Boolean Combinations of $\{\overline{x} \mid \mathsf{p}(\overline{x}) = 0\}$'s and $\{\overline{x} \mid \mathsf{p}(\overline{x}) > 0\}$'s.

## Axiomatizing Mathematical Structures

### Addition, Multiplication and Order of the Reals $\langle \mathbb{R}, +, \cdot, < \rangle$

Tarski: The (First-Order Logical) Theory of the Structure $\langle \mathbb{R}, 0, 1, -, ^{-1}, +, \cdot, < \rangle$ is Decidable and CAN BE AXIOMATIZED As a **Real Closed (Ordered) Field**.

- $x + (y + z) = (x + y) + z$
- $x + y = y + x$
- $x + 0 = x$
- $x + (-x) = 0$
- $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$
- $x < y < z \rightarrow x < z$
- $x < y \rightarrow x + z < y + z$
- $x < y \wedge 0 < z \rightarrow x \cdot z < y \cdot z$
- $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
- $x \cdot y = y \cdot x$
- $x \cdot 1 = x$
- $x \neq 0 \rightarrow x \cdot x^{-1} = 1$
- $0 < 1$
- $x < y \vee x = y \vee y < x$
- $x \not< x$
- $0 < z \rightarrow \exists y (z = y \cdot y)$
- $\forall a_1, \cdots, a_{2n+1} \exists x (x^{2n+1} + \sum_{i=1}^{i=2n+1} a_i x^{i-1} = 0)$ $\quad n \in \mathbb{N}$

## Axiomatizing Mathematical Structures

### Addition and Multiplication of the Real Numbers $\langle\mathbb{R},+,\cdot\rangle$

Tarski: The (First-Order Logical) Theory of the Structure $\langle\mathbb{R},0,1,-,^{-1},+,\cdot\rangle$ is Decidable and Can Be Axiomatized By:

- $x+(y+z)=(x+y)+z$
- $x+y=y+x$
- $x+0=x$
- $x+(-x)=0$
- $x\cdot(y+z)=(x\cdot y)+(x\cdot z)$
- $x^2+y^2+z^2=0\rightarrow x=y=z=0$
- $x\cdot(y\cdot z)=(x\cdot y)\cdot z$
- $x\cdot y=y\cdot x$
- $x\cdot 1=x$
- $x\neq 0\rightarrow x\cdot x^{-1}=1$
- $0\neq 1$
- $\exists y(x=y^2\vee x+y^2=0)$

- $\forall a_1,\cdots,a_{2n+1}\exists x(x^{2n+1}+\sum_{i=1}^{i=2n+1}a_ix^{i-1}=0)$ $\qquad n\in\mathbb{N}$

## Axiomatizing Mathematical Structures

### Addition and Multiplication of Naturals, Integers and Rationals

Can We Axiomatize $\langle\mathbb{N},+,\cdot\rangle$, $\langle\mathbb{Z},+,\cdot\rangle$ or $\langle\mathbb{Q},+,\cdot\rangle$?

$$\begin{array}{ccccccccccccc} \mathbb{N} & \subset & \mathbb{Z} & \subset & \mathbb{Q} & \subset & \mathbb{R}_{\mathrm{Geom.Const.}} & \subset & \mathbb{R}_{\alpha\ell g} & \subset & \mathbb{R} & \subset & \mathbb{C} \\ & & \bigcap & & \bigcap & & \bigcap & & \bigcap & & & & \\ & & \mathbb{Z}[i] & \subset & \mathbb{Q}[i] & \subset & \mathbb{C}_{\mathrm{Geom.Const.}} & \subset & \mathbb{C}_{\alpha\ell g} & & \subset & & \mathbb{C} \end{array}$$

Any Set of Sentences Can Be Regarded As A Set of Axioms

Only When

It Is A Recursively (Computably) Enumerable Set Of Sentences!

Computably Enumerable set $A$: an (input-free) algorithm $\mathcal{P}$ lists all members of $A$; i.e., $A = \mathtt{output}(\mathcal{P})$.

## Computably Enumerable vs. Computably Decidable

• Computably Enumerable set $A$: an (input-free) algorithm $\mathcal{P}$ lists all members of $A$; i.e., $A = \texttt{output}(\mathcal{P})$.

• Computably Decidable set $A$: an algorithm $\mathcal{P}$ decides on any input $x$ whether $x \in A$ (outputs $\texttt{YES}$) or $x \notin A$ (outputs $\texttt{NO}$).

. Post–Kleene's Theorem: A Set is Computably Decidable if and only if Both it and its Complement are Computably Enumerable.

$\therefore$ So, if the theory of a structure $\mathrm{Th}(\mathfrak{A}) = \{\psi \mid \mathfrak{A} \models \psi\}$ is computably enumerable then so is its complement:
$\mathrm{Th}(\mathfrak{A})^{\complement} = \{\theta \mid \mathfrak{A} \not\models \theta\} = \{\theta \mid \mathfrak{A} \models \neg\theta\} = \{\neg\varphi \mid \varphi \in \mathrm{Th}(\mathfrak{A})\}$,
whence it is decidable. Thus

$\mathrm{Th}(\mathfrak{A})$ is decidable $\iff$ $\mathfrak{A}$ is axiomatizable (in a c.e. way)

## Axiomatizability of Mathematical Structures

### Addition and Multiplication $\langle\mathbb{N},+,\cdot\rangle,\langle\mathbb{Z},+,\cdot\rangle,\langle\mathbb{Q},+,\cdot\rangle$

Gödel's First Incompleteness Theorem:

$\mathrm{Th}(\mathbb{N},+,\cdot)$ is Not Computably Enumerable.

An Immediate Corollary:

$\mathrm{Th}(\mathbb{Z},+,\cdot)$ is Not Computably Enumerable.

Because $\mathbb{N}$ is definable in it: for $m\in\mathbb{Z}$ we have

$$m\in\mathbb{N} \iff \exists a,b,c,d(\in\mathbb{Z})\,(m=a^2+b^2+c^2+d^2),$$

by Lagrange's Four Square Theorem.

Neither is $\mathrm{Th}(\mathbb{Q},+,\cdot)$. Since, $\langle\mathbb{Q},+,\cdot\rangle$ can define $\mathbb{Z}$:

J. ROBINSON, *Definability and Decision Problems in Arithmetic*, JSL 14 (1949) 98–114.

B. POONEN, *Characterizing integers among rational numbers with a universal-existential formula*, American Journal of Mathematics 131 (2009) 675–682.

J. KOENIGSMANN, *Defining $\mathbb{Z}$ in $\mathbb{Q}$*, arXiv:1011.3424 [math.NT] (Nov. 2010) (Nov. 2013)

## Axiomatizability of Mathematical Structures

### Addition and Multiplication

$$
\begin{array}{ccccccccccc}
\mathbb{N} & \subset & \mathbb{Z} & \subset & \mathbb{Q} & \subset & \mathbb{R}_{\rm Geom.Const.} & \subset & \mathbb{R}_{\alpha\ell g} & \subset & \mathbb{R} \\
 & & \cap & & \cap & & \cap & & \cap & & \cap \\
 & & \mathbb{Z}[i] & \subset & \mathbb{Q}[i] & \subset & \mathbb{C}_{\rm Geom.Const.} & \subset & \mathbb{C}_{\alpha\ell g} & \subset & \mathbb{C}
\end{array}
$$

| | $\mathbb{N}$ | $\mathbb{Z}$ | $\mathbb{Q}$ | $\mathbb{R}_{\rm G}$ | $\mathbb{R}_{\alpha\ell g}$ | $\mathbb{R}$ |
|---|---|---|---|---|---|---|
| $\{+,\cdot\}$ | $\cancel{\Delta}_1$ | $\cancel{\Delta}_1$ | $\cancel{\Delta}_1$ | ¿? | $\Delta_1$ | $\Delta_1$ |

| | | $\mathbb{Z}[i]$ | $\mathbb{Q}[i]$ | $\mathbb{R}_{\rm G}[i]$ | $\mathbb{R}_{\alpha\ell g}[i]$ | $\mathbb{R}[i]$ |
|---|---|---|---|---|---|---|
| $\{+,\cdot\}$ | | $\cancel{\Delta}_1$ | $\cancel{\Delta}_1$ | ¿? | $\Delta_1$ | $\Delta_1$ |

## Axiomatizability of Mathematical Structures

### Addition and Multiplication of the Complex Numbers $\langle \mathbb{C}, +, \cdot \rangle$

THEORY OF FIELDS WITH ZERO CHARACTERISTIC

- $x + (y + z) = (x + y) + z$
- $x + y = y + x$
- $x + 0 = x$
- $x + (-x) = 0$
- $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$
- $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
- $x \cdot y = y \cdot x$
- $x \cdot 1 = x$
- $x \neq 0 \rightarrow x \cdot x^{-1} = 1$
- $0 \neq 1 + \cdots + 1 = n$

+ SOMETHING ELSE ...

which should also prove

- $\forall a_1, \cdots, a_n \exists x (x^n + \sum_{i=1}^{i=n} a_i x^{i-1} = 0)$ FTA

## Axiomatizing the Field of Complex Numbers

### Another Way ...

$$\texttt{Fields}_0 + \Psi = \mathrm{Th}(\mathbb{C}, +, \cdot)$$

$$\texttt{Fields}_0 + \Psi \vdash \mathrm{FTA}$$

$$\texttt{Fields}_0 + \mathrm{FTA} \vdash \Psi$$

So,

$$\Psi \equiv_{\texttt{Fields}_0} \mathrm{FTA}$$

One Man's Axiom is Another Man's Theorem.

One Man's Theorem is Another Man's Axiom.

story of the Parallel Postulate equivalents

## Axiomatizing or Theoremizing

One Man's Meat is Another Man's Poison.

http://idioms.thefreedictionary.com/

One Man's Loss is Another Man's Gain.

http://dictionary.cambridge.org/

One Man's Trash is Another Man's Treasure.

http://idioms.thefreedictionary.com/

One Man's Ceiling is Another Man's Floor.

http://vimeo.com/55169787

One Man's Magic is Another Man's Engineering.

—Robert A. Heinlein

## Axiomatizing or Theoremizing

One Man's Mistake is Another Man's Opportunity.

—Steven Brust

One Man's Home is Another Man's Uranium Dump.

http://mg.co.za/

One Man's Hate is Another Man's Faith.

http://fullcomment.nationalpost.com/

One Man's Terrorist is Another Man's Freedom Fighter. ...

—Kayode Olatunbosun (Author House 2011)

One Man's Tall is Another Man's Small: ...

Health Economics 23:7 (2014) 776–791

## Algebraic Geometry vs. Mathematical Logic

One Man's Theorem is Another Man's Principle.

Tarski & Chevalley:

$$\exists x\big(\bigwedge_i p_i(x,\bar{y})=0 \wedge \bigwedge_j q_j(x,\bar{y})\neq 0\big) \equiv_{\mathbb{C}}$$
$$\equiv_{\mathbb{C}} \bigvee_{l,n}\big(\bigwedge_k P_{k,l}(\bar{y})=0 \wedge \bigwedge_m Q_{m,n}(\bar{y})\neq 0\big)$$

Tarski & Seidenberg:

$$\exists x\big(\bigwedge_i p_i(x,\bar{y})=0 \wedge \bigwedge_j q_j(x,\bar{y})> 0\big) \equiv_{\mathbb{R}}$$
$$\equiv_{\mathbb{R}} \bigvee_{l,n}\big(\bigwedge_k P_{k,l}(\bar{y})=0 \wedge \bigwedge_m Q_{m,n}(\bar{y})> 0\big)$$

## Axiomatizing the Field of Real Numbers

### Another Way ...

So, any proof of $\texttt{Fields}_0 \vdash_{\mathbb{C}} \text{FTA}$ should give away another axiomatization of $\langle \mathbb{C}, +, \cdot \rangle$. But most of the proofs are in $\mathbb{R}$.

Another Way of Axiomatizing the Real Field?

THEORY OF FORMALLY REAL FIELDS WITH SQUARE ROOTS

- $x + (y + z) = (x + y) + z$
- $x + y = y + x$
- $x + 0 = x$
- $x + (-x) = 0$
- $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$
- $x^2 + y^2 + z^2 = 0 \rightarrow x = y = z = 0$
- $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
- $x \cdot y = y \cdot x$
- $x \cdot 1 = x$
- $x \neq 0 \rightarrow x \cdot x^{-1} = 1$
- $0 \neq 1$
- $\exists y (x = y^2 \vee x + y^2 = 0)$

+ SOMETHING ELSE ...

## Axiomatizing the Field of Real Numbers

Another Way ...

$\mathtt{Fields}_{\sqrt{+}} + \Psi = \mathrm{Th}(\mathbb{R}, +, \cdot)$ So,

$$\Psi \equiv_{\mathtt{Fields}_{\sqrt{+}}} \mathrm{FTA}_{\mathtt{odd}} = \Big\{\forall\bar{a}\exists x(x^{2n+1} + \sum_{i=1}^{2n+1} a_i x^{i-1} = 0)\Big\}_n$$

Suggestions:

$\mathrm{FTA}_{\mathbb{R}} = \forall\bar{a}\exists\bar{b},\bar{c}\forall x\Big((x^{2n} + \sum_{i=1}^{2n} a_i x^{i-1}) = \prod_{j=1}^{n}(x^2 + b_j x + c_j)\Big)$

$\mathrm{IVT} = \forall P\forall u, v\exists x\Big[u<v \wedge P(u)\cdot P(v)<0 \longrightarrow u<x<v \wedge P(x)=0\Big]$

Intermediate Value Theorem $\forall P = \forall\bar{a},\ P(y) = y^m + \sum_{i=1}^{m} a_i y^{i-1}$

## Alternative Axiomatizations for the Field of Real Numbers

Three Beautiful Proofs

### Theorem

$\texttt{Fields}_{\sqrt{+}} + \mathrm{FTA}_{\mathbb{R}} \vdash \mathrm{FTA}_{\texttt{odd}}$

### Proof.

$\forall\bar{a}\exists\bar{b}\exists\bar{c}\Big[\big(x^{2n+2} + \sum_{i=1}^{2n+1} a_i x^i\big) = \prod_{j=1}^{n+1}(x^2 + b_j x + c_j)\Big]$. Since, $\prod_{j=1}^{n+1} c_j = 0$, for some $j$, $c_j = 0$. Put $c_{n+1} = 0$. Whence $x \cdot \big(x^{2n+1} + \sum_{i=1}^{2n+1} a_i x^{i-1}\big) = x \cdot (x + b_{n+1}) \cdot \prod_{j=1}^{n}(x^2 + b_j x + c_j)$. Thus $(-b_{n+1})^{2n+1} + \sum_{i=1}^{2n+1} a_i(-b_{n+1})^{i-1} = 0$. □

### Question

*A Nice (First–Order) Proof For* $\texttt{Fields}_{\sqrt{+}} + \mathrm{FTA}_{\texttt{odd}} \vdash \mathrm{FTA}_{\mathbb{R}}$*?*

## Theorem

$\texttt{Fields}_{\sqrt{+}} + \mathrm{FTA}_{\mathbb{R}} \vdash \mathrm{IVT}$

## Proof.

For $P(y) = \sum_{i=1}^{m} a_i y^{i-1}$ with $u < v$ and $P(u) \cdot P(v) < 0$, put $Q(y) = \frac{1}{P(u)}(1+y^2)^m P(u + \frac{v-u}{1+y^2})$. Then $Q(y) = y^{2m} + R(y^2)$ with $\deg(R) < m$ and $Q(0) = \frac{P(v)}{P(u)} = \frac{P(u)P(v)}{P(u)^2} < 0$. For some $\bar{b}$ and $\bar{c}$ we have $Q(y) = \prod_{j=1}^{m}(y^2 + b_j y + c_j)$. Then $\prod_{j=1}^{m} c_j < 0$ and so some $c_j < 0$. Whence, $Q(\mathfrak{z}) = 0$ for $\mathfrak{z} = \frac{1}{2}(-b_j + \sqrt{b_j^2 - 4c_j})$ and for $\mathfrak{x} = u + \frac{v-u}{1+\mathfrak{z}^2}$ we have $u < \mathfrak{x} < v$ and $P(\mathfrak{x}) = 0$. $\square$

## Question

*A Nice (First–Order) Proof For* $\texttt{Fields}_{\sqrt{+}} + \mathrm{IVT} \vdash \mathrm{FTA}_{\mathbb{R}}$*?*

Indeed, we have the following classical result:

**Theorem**

$\texttt{Fields}_{\sqrt{+}} + \mathrm{IVT} \vdash \mathrm{FTA}_{\texttt{odd}}$

**Proof.**

For $P(x) = x^{2n+1} + \sum_{i=1}^{2n+1} a_i x^{i-1}$ let $v = 1 + \sum_i |a_i|$ and $u = -v$.
Then $u \leqslant -1 < 0 < 1 \leqslant v$, so $|u|, |v| \geqslant 1$ and by $u + \sum_i |a_i| = -1$,
$P(u) = u^{2n+1} + \sum_i a_i u^{i-1} \leqslant u^{2n+1} + \sum_i |a_i||u|^{i-1} \leqslant$
$u^{2n+1} + \sum_i |a_i| u^{2n} = u^{2n}(u + \sum_i |a_i|) = -u^{2n} < 0$; also
$P(v) = v^{2n+1} + \sum_i a_i v^{i-1} \geqslant v^{2n+1} - \sum_i |a_i||v|^{i-1} \geqslant$
$v^{2n+1} - \sum_i |a_i| v^{2n} = v^{2n}(v - \sum_i |a_i|) = v^{2n} > 0.$ $\square$

**Question**

*A Nice (First–Order) Proof For* $\texttt{Fields}_{\sqrt{+}} + \mathrm{FTA}_{\texttt{odd}} \vdash \mathrm{IVT}$?

**The Fundamental Theorem of Algebra**

is then really FUNDAMENTAL

For Algebra, Analysis on Polynomials, Algebraic Geometry, First–Order Logical Axiomatization of Addition and Multiplication in Real (algebraic) and (algebraic) Complex Numbers, etc.

**The Fundamental Theorem of Algebra**

equivalent of other axiomatizations for reals

- Order Completeness of $\mathbb{R}$
- Induction on $\mathbb{R}$
- etc.

## Axiomatizing the Field of Real Numbers

### Another Way ... ?

Order Completeness of $\mathbb{R}$: (in second order logic)

For any $X \subseteq \mathbb{R}$, if $X \neq \emptyset$ and $\exists u[\forall x \in X : x \leqslant u]$ then $\exists s[s = \sup X]$ i.e., $\forall y[(\forall x \in X : x \leqslant y) \longleftrightarrow s \leqslant y]$.

First–Order Scheme: (OrdCom)

$$\exists x \varphi(x) \wedge \exists u[\forall x(\varphi(x) \rightarrow x \leqslant u)] \longrightarrow \exists s \forall y[\forall x(\varphi(x) \rightarrow x \leqslant y) \leftrightarrow s \leqslant y]$$

Classical Real Analysis:

$\texttt{OrderedFields} + \mathrm{OrdCom} \vdash \texttt{Fields}_{\sqrt{+}}$ $(\forall x \exists y[0 < x \rightarrow x = y^2])$.

$\varphi(z) \equiv z^2 < x$

$\texttt{OrderedFields} + \mathrm{IVT} \vdash \texttt{Fields}_{\sqrt{+}}$ $(\forall x \exists y[0 < x \rightarrow x = y^2])$.

$P(z) = z^2 - x$

$\texttt{OrderedFields} + \mathrm{OrdCom} \vdash \mathrm{IVT}$.

$\varphi(z) \equiv u < z < v \wedge P(z)P(v) < 0$

## Question

*A Nice* $1^{\rm st}$ *Order Proof of* $\texttt{OrderedFields} + \mathrm{OrdCom} \vdash \mathrm{FTA}_{\mathbb{R}}$ ?

*A Nice* $1^{\rm st}$ *Order Proof of* $\texttt{Fields}_{\sqrt{+}} + \mathrm{FTA}_{\mathbb{R}} \vdash \mathrm{OrdCom}$ ?

*A Nice* $1^{\rm st}$ *Order Proof of* $\texttt{OrderedFields} + \mathrm{IVT} \vdash \mathrm{OrdCom}$ ?

Dedekind Completeness of $\mathbb{R}$: (`in second order logic`)

For any $\emptyset \neq X, Y \subseteq \mathbb{R}$, if $X \prec Y$ then $\exists \nu [X \preccurlyeq \nu \preccurlyeq Y]$

First–Order Scheme: ($\mathrm{Ddk\text{--}Com}$)

$$\exists x \varphi(x) \wedge \exists y \psi(y) \wedge \forall x, y (\varphi(x) \wedge \psi(y) \rightarrow x < y) \longrightarrow$$
$$\longrightarrow \exists \nu \forall x, y \big( [\varphi(x) \rightarrow x \leqslant \nu] \wedge [\psi(y) \rightarrow \nu \leqslant y] \big)$$

$\texttt{OrderedFields} + \mathrm{Ddk\text{--}Com} \vdash \mathrm{OrdCom}$

$$\psi(y) \equiv \forall x [\varphi(x) \rightarrow x \leqslant y]$$

$\texttt{OrderedFields} + \mathrm{OrdCom} \vdash \mathrm{Ddk\text{--}Com}$

$$\sup_x \varphi(x) \preccurlyeq \psi$$

## Axiomatizing the Field of Real Numbers

### Continuous Induction

PETE L. CLARK, The Instructor's Guide to Real Induction

http://www.math.uga.edu/~pete/instructors_guide_shorter.pdf

"The earliest instance ... 1919 note of Y.R. Chao ..."

Y.R. CHAO, A note on "Continuous mathematical induction", *Bull Amer Math Soc* 26:1.

"A literature search turned up the following papers, each of which introduces some form of 'continuous induction', in many cases with no reference to past precedent:"

http://www.math.uga.edu/~pete/Kalantari07.pdf

$\boxed{\mathrm{Ind}_{\mathbb{R}}}$ $\exists x\varphi(x)\wedge\forall x,y[y<x\wedge\varphi(x)\rightarrow\varphi(y)]\wedge\forall x\exists y[\varphi(x)\rightarrow x<y\wedge\varphi(y)] \longrightarrow \forall x\varphi(x).$

and the story goes on ...

## Axiomatizing the Field of Real Numbers

after constructing it

A Fast(er) Way of Constructing Reals (and short proving its completeness):

- Build $\mathbb{N}^+ = \{1, 2, 3, \cdots\}$ by Induction $\langle 1, \mathfrak{s}\rangle$ $\qquad \mathbb{N}^+ = \langle 1\rangle_{\mathfrak{s}}$

$x + 1 = \mathfrak{s}(x), \;\; x + \mathfrak{s}(y) = (x + y); \qquad x \cdot 1 = x, \;\; x \cdot \mathfrak{s}(y) = (x \cdot y) + y.$

- Build $\mathbb{Q}^+$ by $[\mathbb{N}^+/\mathbb{N}^+]_{\approx}$ where $(a/b) \approx (p/q) \iff a \cdot q = b \cdot p$

$(a/b) + (p/q) = (a \cdot q + b \cdot p)/(p \cdot q) \qquad (a/b) \cdot (p/q) = (a \cdot p)/(b \cdot q)$

- $x < y \iff \exists z[z + x = y]$
- Build $\mathbb{R}^+$ as nonempty, downward closed, and bounded above subsets of $\mathbb{Q}^+$ without maximum elements:

IWASE ZJUÑCI, Rapid (Quick) [Fast] Construction of Real Numbers by Half-Cuts

http://www5a.biglobe.ne.jp/∼iwase47/HalfCuts4.pdf (HalfCuts5.pdf) [HalfCuts6.pdf]

- Adding $0$ is easy and adding negatives can be done either by (i) introducing the minus function $-$ or (ii) constructing the equivalence classes of subtractions (the usual way of constructing $\mathbb{Z}$ from $\mathbb{N}$).

# Thank You!

Thanks to

The Participants ....................For Listening...

and

The Organizers ....For Taking Care of Everything...

SaeedSalehi.ir