Hello!

Gödel's Incompleteness Theorem: Constructivity of Its Various Proofs

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Tutorial I: Constructive Proofs
30 May 2016

Outline

- Tutorial I:
  Constructive Proofs — 30 May 2016
- Tutorial II:
  Gödel's Incompleteness Theorem — 30 May 2016
- Tutorial III:
  Constructivity of Proofs for Gödel's Theorem — 31 May 2016

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Why Constructivism?

G.J. Chaitin, *Thinking about Gödel & Turing* (W.S. 2007) p. 97

So in the end it wasn't Gödel, it wasn't Turing, [...] that are making mathematics go in an experimental mathematics direction, in a quasi-empirical direction. The reason that mathematicians are changing their working habits is the computer. I think it's an excellent joke! (It's also funny that of the three old schools of mathematical philosophy, logicist, formalist, and intuitionist, the most neglected was Brouwer, who had a constructivist attitude years before the computer gave a tremendous impulse to constructivism.)

What is Constructivism?

D. BRIDGES, Constructive Mathematics, *Stanford Encyclopedia of Philosophy* (1997, 2013) http://plato.stanford.edu/entries/mathematics-constructive/

Constructive mathematics is distinguished from its traditional counterpart, classical mathematics, by the strict interpretation of the phrase "there exists" as "we can construct".

⋮⋮⋮

[It is] developing mathematics in such a way that when a theorem asserts the existence of an object x with a property P, then the proof of the theorem embodies algorithms for constructing x and for demonstrating, by whatever calculations are necessary, that x has the property P.

A Simple Example

A Theorem with Constructive and Nonconstructive Proofs

A constructive (nonconstructive) proof shows the existence of an object by presenting (respectively, without presenting) the object. From a logical point of view, a constructive (nonconstructive) proof does not use (respectively, uses) the law of the excluded middle.

The discussion of constructive versus nonconstructive proofs is very common in mathematical logic and philosophy. To illustrate this discussion, it is convenient to have some *very simple* examples of theorems with both constructive and nonconstructive proofs. Unfortunately, there seems to be a shortage of such examples. We present here a new example.

Theorem. *Let c be an arbitrary real constant. The equation $c^2x^2 - (c^2 + c)x + c = 0$ in x has a real solution.*

Nonconstructive proof. By the law of the excluded middle, we have $c = 0$ or $c \neq 0$.

- Case $c = 0$: $x = 0$ (or any x) is a solution.
- Case $c \neq 0$: $x = 1/c$ is a solution.

(This proof is nonconstructive because it does not present a solution, that is, it does not decide between the two cases as the equality $c = 0$ is undecidable.) ∎

The American Mathematical Monthly, vol. 120 no. 6 (2013) page 536.

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Constructive proof. We have that $x = 1$ is a solution. (This proof is constructive because it presents a solution.) ∎

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Constructive Proofs $\rightsquigarrow$ Algorithms

Theorem (The Intermediate Value Theorem)

For any polynomial (in general, continuous) $f:\mathbb{R}\to\mathbb{R}$ if $f(a)f(b)<0$ then for some $c\in[a,b]$ we have $f(c)=0$.

Non-Constructive Proof.

Let $c=\sup\{x\in[a,b] : f(a)f(x)>0\}$ (the largest root of f in $[a,b]$) or $c=\inf\{x\in[a,b] : f(b)f(x)>0\}$ (the smallest). ❑

Constructive Proof.

Define $[a_n,b_n]$'s by induction: $[a_0,b_0]=[a,b]$, and

$$[a_{n+1},b_{n+1}]=\begin{cases}[a_n,\frac{a_n+b_n}{2}] & \text{if } f(a_n)f(\frac{a_n+b_n}{2})<0,\\ [\frac{a_n+b_n}{2},b_n] & \text{if } f(a_n)f(\frac{a_n+b_n}{2})>0,\\ \{\frac{a_n+b_n}{2}\} & \text{if } f(a_n)f(\frac{a_n+b_n}{2})=0;\end{cases}$$

and let $c=\lim_n a_n$ (or $\lim_n b_n$). ❑

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Another Example

Web-Page of David Duncan at Michigan State University

http://users.math.msu.edu/users/duncan42/Recitation7.pdf

Theorem (The Archemidean Property of the Rationals)

$\forall r \in \mathbb{Q}\, \exists n \in \mathbb{N} : r < n.$

Non-Constructive Proof.

If for $r = \frac{p}{q} \in \mathbb{Q}$, we have $\forall n \in \mathbb{N} : n \leqslant r$, then we can assume that $p, q \in \mathbb{N} - \{0\}$, and so $\frac{p}{q} > p$ whence $0 < q < 1$, contradiction! ❑

Constructive Proof.

Write $r = \frac{p}{q}$ with $p \in \mathbb{Z}, q \in \mathbb{N}$. Now, from $1 \leqslant q$ we have $0 < \frac{1}{q} \leqslant 1$ and so $r = \frac{p}{q} \leqslant |p| < |p| + 1 (= n)$. ❑

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The Most Well-Known Example (I)

Theorem (Some Irrational Power of an Irrational Could Be Rational)

There are irrational numbers a, b such that a^b is rational.

Non-Constructive Proof.

If $\sqrt{2}^{\sqrt{2}}$ is rational then we are done with $a = b = \sqrt{2}$ (below) otherwise $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = 2$ proves the theorem with $a = \sqrt{2}^{\sqrt{2}}, b = \sqrt{2}$. ❑

Proof (of the irrationality of $\sqrt{2}$).

If $\sqrt{2} = \frac{p}{q}$ then $p^2 = 2q^2$, but the exponent of 2 in the unique prime factorization of p^2 is even while it is odd in $2q^2$, contradiction! ❑

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By GELFOND-SCHNEIDER theorem $\sqrt{2}^{\sqrt{2}}$ is irrational.

Does this theorem have a *Constructive Proof*?

Constructive Proof.

For $a = \sqrt{2}, b = 2\log_2 3$ we have

$$a^b = (\sqrt{2})^{2\log_2 3} = 2^{\log_2 3} = 3. \qquad \square$$

Proof (of the irrationality of $\log_2 3$).

If $\log_2 3 = \frac{p}{q}$ with $p, q \in \mathbb{N} - \{0\}$, then $q \log_2 3 = p$ and so $\log_2 3^q = p$ whence $3^q = 2^p$, contradiction! $\square$

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Even More Constructive Proofs

A Constructive Proof for the irrationality of $\sqrt{2}$.

By JOSEPH LIOUVILLE's theorem for any $p, q \in \mathbb{N}^+$ we have

$$|\sqrt{2} - \frac{p}{q}| > \frac{C}{q^2} > 0$$

for some computable (from p, q) constant C. ❑

Joseph Liouville 1809—1882 a famous French mathematician

A Constructive Proof for the irrationality of $\log_2 3$.

By ALAN BAKER's theorem for any $p, q \in \mathbb{N}^+$ we have

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Alan Baker 1939—¿ English mathematician (Fields Medalist in 1970)

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Alan Baker 1939—¿ English mathematician (Fields Medalist in 1970)

Even More Constructive Proofs

A Constructive Proof for the irrationality of $\sqrt{2}$.

By JOSEPH LIOUVILLE's theorem for any $p, q \in \mathbb{N}^+$ we have

$$|\sqrt{2} - \frac{p}{q}| > \frac{C}{q^2} > 0$$

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Joseph Liouville 1809—1882 a famous French mathematician

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One More Example (1)

Definition (Outgoing Set)

In a directed graph $\langle V; E\rangle$ (where $E\subseteq V^2$) *outgoing set* of a vertex $a\in V$ is $\{x\in V \mid aEx\}$. ✧

Example: In the directed graph

$$\begin{tikzcd}
 & b \arrow[d] & c \arrow[l] \\
x \arrow[ur] \arrow[r, shift right] & a \arrow[l, shift right] \arrow[r, shift left] \arrow[loop below] & y \arrow[l, shift left] \arrow[u]
\end{tikzcd}$$

we have $x\mapsto\{b,a\}$, $a\mapsto\{x,a,y\}$, $b\mapsto\{a\}$, $y\mapsto\{a,c\}$, $c\mapsto\{b\}$.

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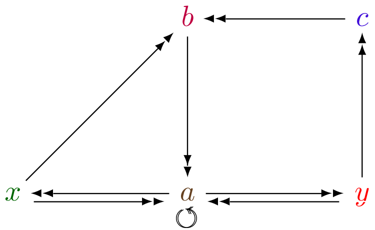

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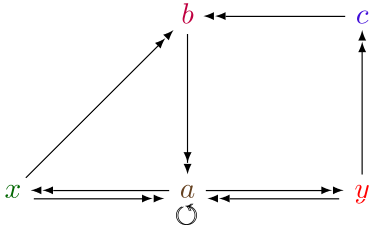

we have $x \mapsto \{b, a\}, a \mapsto \{x, a, y\}, b \mapsto \{a\}, y \mapsto \{a, c\}, c \mapsto \{b\}$.

One More Example (2)

Theorem

In any (finite) *directed graph, there exists a set of vertices which is not the outgoing set of any vertex.*

Lemma

(i) *Any set with n elements has 2^n subsets.*
(ii) *For any $n \in \mathbb{N}$ we have $2^n > n$.*

Proof.

By induction on n: trivial for $n = 0, 1$.
(i) for $n+1$: if $A = B \cup \{\alpha\}$ with $\alpha \notin B$ then every subset of A is either (1) a subset of B or (2) a subset of B with α. So, the number of the subsets of A is the double number of the subsets of B.
(ii) for $n+1$: $2^{n+1} = 2 \cdot 2^n >_{(i.h.)} 2 \cdot n \geqslant n+1$ (for $n \geqslant 1$). ❑

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For any directed graph with n nodes we have 2^n (sub)sets of nodes [by Lemma(i)] and at most n outgoing sets. Thus [from Lemma(ii)] there must exist some set of nodes which is not outgoing. ❑

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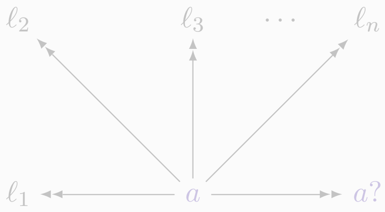
Theorem (G. CANTOR)

In any directed graph, there exists a set of vertices which is not the outgoing set of any vertex.

Constructive Proof.

Let $\texttt{LoopLess} = \{x \in V \mid x \not\!E x\}$.

If $\{\ell_1, \ell_2, \ell_3, \cdots\} = \texttt{LoopLess} = \text{Outgoing}(a) = \{x \mid aEx\}$



then $a \not\!E a \longleftrightarrow a \in \texttt{LoopLess} \longleftrightarrow a \in \{x \mid aEx\} \longleftrightarrow aEa\,!$ $\square$

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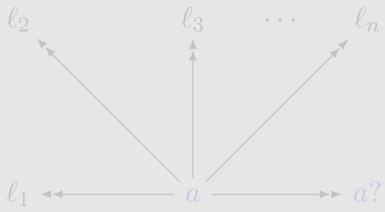

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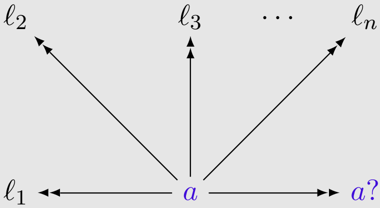

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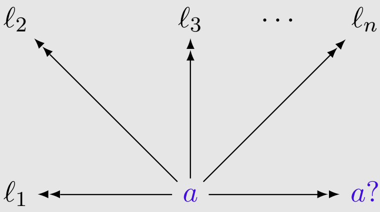

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One More Example (5)

More Constructive (Diagonal) Proofs.

For any injective $\mathfrak{g}:V\to V$ let $D_{\mathfrak{g}}=\{\mathfrak{g}(x)\mid x\not E\mathfrak{g}(x)\}$. For any $a\in V$ we have

$$\mathfrak{g}(a)\in D_{\mathfrak{g}} \longleftrightarrow \exists x.\mathfrak{g}(a)=\mathfrak{g}(x)\,\&\,x\not E\mathfrak{g}(x)$$
$$\longleftrightarrow a\not E\mathfrak{g}(a) \longleftrightarrow \mathfrak{g}(a)\notin \mathrm{Outgoing}(a),$$

and so $D_{\mathfrak{g}}$ differs from every $\mathrm{Outgoing}(a)$ set (at $\mathfrak{g}(a)$). ❑

A New Theorem:

EVERY SUCH SET (different from any outgoing set) is *Constructed* as above for some suitable (not necessarily injective) function $\mathfrak{g}$.

So, its every constructive proof is a diagonal argument.

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See You Later

Lots of Open Problems &

A Nice Question to Ask at the End of Lectures (to hide sleepiness):

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To Be Continued ...

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Thank You!

Thanks to

The Participants .... For Listening ···

and

The Organizers — For Taking Care of Everything ···

SaeedSalehi.ir

Hello!

Gödel's Incompleteness Theorem: Constructivity of Its Various Proofs

Saeed Salehi

University of Tabriz & IPM

http://SaeedSalehi.ir/

SWAMPLANDIA 2016, Ghent University
Tutorial II: Gödel's Incompleteness Theorem
30 May 2016

Outline

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A Conversation At The End Of A Lecture

question Does Your Theorem Have A Constructive Proof?

answer YES / NO / I Don't Know

question (if YES) Do You Know Its (Computational) Complexity?

question (if NO) Have Your Proved It?
(the it can never have a constructive proof?)

answer ... Oh ... Well ... YES / NO

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Chaitin-Kolmogorov Complexity (1)

Definition (Information-Theoretic Complexity)

The (descriptive) COMPLEXITY of an *object* is the least (minimum) *size* of a process (program) that results (produces/outputs) it. ✧

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$$\begin{array}{ccc} & & \mathbb{N} \\ & \overset{\text{COMPLEXITY}}{\nearrow} & \uparrow \textit{size} \\ \textit{Obj.} & \xleftarrow[\text{result}]{} & \textit{Proc.} \end{array}$$

$$\text{COMPLEXITY}(\textit{object}) = \min \textit{size}[\text{result}^{-1}(\textit{object})]$$

Chaitin-Kolmogorov Complexity (2)

$$\begin{tikzcd} & \mathbb{N} \\ Obj. \arrow[ur, dashed, "\text{COMPLEXITY}"] & Proc. \arrow[l, "\text{result}"'] \arrow[u, "size"'] \end{tikzcd}$$

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Example (A Simple One)

Let $Obj = \mathbb{N}$, $Proc = \langle \mathfrak{c}_0, \mathfrak{c}_1, \cdots \rangle = \mathbb{N}$, $\text{result}(\mathfrak{c}_i) = \mathfrak{c}_i$, $size(\mathfrak{c}_i) = i$.

Then $\text{COMPLEXITY}(n) = \min\{i \mid (\mathfrak{c}_i = n)\}$.

If $Proc = \langle \underbrace{0}_{1}, \underbrace{1,1}_{2}, \underbrace{2,2,2}_{3}, \underbrace{3,3,3,3}_{4}, \underbrace{4,4,4,4,4}_{5}, \cdots \rangle$ then

$\underbrace{\mathcal{C}(0)=0}_{\mathfrak{c}_0=0}, \underbrace{\mathcal{C}(1)=1}_{\mathfrak{c}_1=1}, \underbrace{\mathcal{C}(2)=3}_{\mathfrak{c}_3=2}, \underbrace{\mathcal{C}(3)=6}_{\mathfrak{c}_6=3}, \cdots, \mathcal{C}(n) = \frac{n(n+1)}{2}, \cdots.$ ✧

Some Computability Theory

Convention (Classic Computability-Theoretic Notation)

Enumerate all the single-input computable (partial) functions $\mathbb{N}\to\mathbb{N}$ *as*

$$\varphi_0, \varphi_1, \varphi_2, \cdots .$$

Denote the universal (computable) function by $\Phi(x,y)=\varphi_x(y)$.

There exists a computable (partial) binary function $\Phi: \mathbb{N}^2 \to \mathbb{N}$ such that for any computable (partial) unary function $f: \mathbb{N}\to\mathbb{N}$ there is some $e\in\mathbb{N}$ such that $f(x)=\Phi(e,x)$. ✧

Example (Recursion–Theoretic)

Let $Obj=\mathbb{N}$, $Proc=\{\varphi_0,\varphi_1,\varphi_2,\cdots\}$, $\mathrm{result}(\varphi_i)=\varphi_i(0)$, and $size(\varphi_i)=i$. Then (also with $Proc=\langle\varphi_0(0),\varphi_1(0),\varphi_2(0),\cdots\rangle$) $\text{COMPLEXITY}(n)=\min\{i \mid (\varphi_i(0)=n)\}=\mathscr{K}(n)$. ✧

(Chaitin–)$\mathscr{K}$olmogorov Complexity

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(Chaitin–)$\mathscr{K}$olmogorov Complexity

Chaitin-Kolmogorov Complexity (3)

$$\begin{tikzcd} & \mathbb{N} \\ Obj. \arrow[ur, dashed, "\textsc{complexity}"] & Proc. \arrow[l, "\text{result}"] \arrow[u, "size"'] \end{tikzcd}$$

$$\textsc{complexity}(object) = \min size[\text{result}^{-1}(object)]$$

Lemma (The Main Lemma)

If the set Obj of objects is infinite and for any $n \in \mathbb{N}$ the set $size^{-1}(n)$ of processes with size n is finite, then for any $m \in \mathbb{N}$ there exists some object ℓ such that $\textsc{complexity}(\ell) > m$.

Non-Constructive Proof.

The set $\bigcup_{i \leqslant m} size^{-1}(i)$ is finite and so is the set $\{\alpha \in Obj \mid \textsc{complexity}(\alpha) \leqslant m\} = \bigcup_{i \leqslant m} \text{result}[size^{-1}(i)]$. □

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$\mathcal{C}(n) = \frac{n(n+1)}{2}$ and so $\mathcal{C}(m+1) > m$ for any $m \in \mathbb{N}$. ✧

Example (Kolmogorov Complexity)

Is there a computable function f with $\forall m \in \mathbb{N}\ \mathscr{K}(f(m)) > m$? ✧

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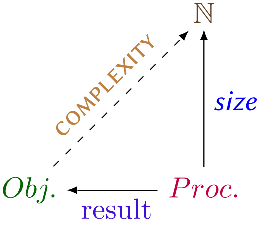

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A Non-Constructive Theorem

Theorem (Non-Constructivity of the Main Lemma)

There is no computable function f such that $\forall m \in \mathbb{N}\ \mathscr{K}\big(f(m)\big) > m$.

Berry's Paradox:

The Smallest Number Not Outputable by Program-*Size* of $\leqslant \cdots$

Proof.

For any f by Kleene's (2nd) Recursion (fixed-point) Theorem there exists some e such that $\varphi_e(0) = f(e)$, thus $\mathscr{K}\big(f(e)\big) \leqslant e$! ❑

A Cornerstone of Computability Theory

Kleene 's Second Recursion Theorem: For any computable $f : \mathbb{N} \to \mathbb{N}$ there exists some $e \in \mathbb{N}$ such that $\varphi_e(0) = f(e)$.

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Chaitin-Kolmogorov Complexity (4)

Corollary (Uncomputability of $\mathscr{K}$)

The Kolmogorov Complexity is not computable.

Proof.

Otherwise, $f(x) = \min\{z \mid \mathscr{K}(z) > x\}$ which satisfies $\forall x : \mathscr{K}\big(f(x)\big) > x$ would be computable by this algorithm:

```
input x
put y := 0
while K(y) ≤ x do
    {y := y + 1}
print y
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This would contradict The Main Lemma. ❑

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Some More Computability Theory (i)

Definition (Computably Decidable)

A set $A \subseteq \mathbb{N}$ with an algorithm $\mathcal{P}$ decides on any input x whether $x \in A$ (outputs YES) or $x \notin A$ (outputs NO).

$$\xrightarrow{\texttt{input: } x\in\mathbb{N}} \boxed{\text{Algorithm}} \xrightarrow{\texttt{output:}} \begin{cases} \texttt{YES} & \text{if } x \in A \\ \texttt{NO} & \text{if } x \notin A \end{cases}$$

Algorithm: single–input (natural number), Boolean–output (1/0). ✧

Definition (Semi–Decidable)

A set $A \subseteq \mathbb{N}$ with an algorithm $\mathcal{P}$ halts on any input x if and only if $x \in A$ (and does not halt if and only if $x \notin A$).

$$\xrightarrow{\texttt{input: } x\in\mathbb{N}} \boxed{\text{Algorithm}} \longrightarrow \begin{cases} \downarrow \texttt{halt} & \text{if } x \in A \\ \uparrow \texttt{loop} & \text{if } x \notin A \end{cases}$$

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Some More Computability Theory (ii)

Example

Almost all the sets of natural numbers that we know:

- every finite set
- $\{0, 3, 6, 9, \cdots, 3k, \cdots\}$
- $\{0, 1, 4, 9, 16, 25, \cdots, k^2, \cdots\}$
- $\{2, 3, 5, 7, 11, 13, \cdots, \mathfrak{prime}, \cdots\}$ ✧

Theorem (Decidability ≡ SemiDecidability + Co-SemiDecidability)

A set is decidable iff it and its complement are both semidecidable.

Proof.

If $\mathcal{P}$ semidecides A and $\mathcal{Q}$ semidecides $\overline{A}$ then for deciding A, on any input, run $\mathcal{P}$ and $\mathcal{Q}$ in parallel (a step of each in turn) and if $\mathcal{P}$ halts then print YES and if $\mathcal{Q}$ halts then print NO. ❑

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A Semi-Decidable But Un-Decidable Set

Theorem ($2^{\aleph_0} > \aleph_0$)

There exists a semi-decidable but undecidable set.

	0	1	2	3	4	5	$\cdots$
φ_0	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\cdots$
φ_1	$\downarrow$	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\uparrow$	$\cdots$
φ_2	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\cdots$
φ_3	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\downarrow$	$\downarrow$	$\cdots$
φ_4	$\downarrow$	$\downarrow$	$\uparrow$	$\uparrow$	$\downarrow$	$\downarrow$	$\cdots$
φ_5	$\uparrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\overline{K}$	X	1	2	3	X	X	$\cdots$
K	0	X	X	X	4	5	$\cdots$

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φ_4	$\downarrow$	$\downarrow$	$\uparrow$	$\uparrow$	$\downarrow$	$\downarrow$	$\cdots$
φ_5	$\uparrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\overline{K}$	X	1	2	3	X	X	$\cdots$
K	0	X	X	X	4	5	$\cdots$

A Semi-Decidable But Un-Decidable Set

Theorem ($2^{\aleph_0} > \aleph_0$)

There exists a semi-decidable but undecidable set.

	0	1	2	3	4	5	$\cdots$
φ_0	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\cdots$
φ_1	$\downarrow$	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\uparrow$	$\cdots$
φ_2	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\cdots$
φ_3	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\downarrow$	$\downarrow$	$\cdots$
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Theorem (A Diagonal Argument)

There exists a semi-decidable but undecidable set.

(Constructive) Proof.

If $\overline{K} = \{n \in \mathbb{N} \mid \varphi_n(n)\uparrow\}$ were semi-decidable by (say) φ_k, then

$$\varphi_x(x)\uparrow \iff x \in \overline{K} \iff \varphi_k(x)\downarrow$$

so, for $x = k$,

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contradiction!

Whence, $\overline{K}$, and also $K = \{n \in \mathbb{N} \mid \varphi_n(n)\downarrow\}$, is undecidable.

But the set $K = \{n \in \mathbb{N} \mid \varphi_n(n)\downarrow\}$ is semi-decidable by the (computable) function $n \mapsto \Phi(n, n)$ since,

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Computability Theory in Mathematical Logic

The set of PROOFS of an *Axiomatizable Theory* must be **Decidable**.

The **decidability** of its *set of axioms* suffices (and is necessary).

Proposition (Axioms ∈ Dec. ⟹ Proofs ∈ Dec. & Theorems ∈ SeDec.)

If the set of axioms of a theory is decidable, then the set of its proofs is decidable, and the set of its theorems is semi-decidable.

Proof.

If T is decidable, then the set of sequences $\langle\psi_0, \psi_1, \cdots, \psi_n\rangle$ with

- each ψ_i is either a logical axiom or a member of T, or
- each ψ_i results from some previous ones by an inference rule,

is decidable. Now, a formula ψ is a theorem of T if and only if one can find such a sequence with $\psi_n = \psi$. ❑

Proof Search Algorithm AUTOMATED THEOREM PROVING

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Gödel's First Incompleteness Theorem

Follows from (and in fact is equivalent to)

the existence of a semi-decidable but un-decidable set:

Theorem (Gödel's First Incompleteness Theorem—Semantic Form)

No semi-decidable and sound theory can be complete.

Kleene's Proof.

For a semi-decidable and undecidable set A (such that $\overline{A}$ is not semi-decidable) let $\overline{A}_T = \{n \in \mathbb{N} \mid T \vdash \text{“}n \notin A\text{”}\}$.

Then, by the soundness of T we have $\overline{A}_T \subseteq \overline{A}$,

but $\overline{A}_T$ is semi-decidable $[n \mapsto \text{Proof-Search}_T(n \notin A)]$ and $\overline{A}$ isn't.

So, there must be some $\mathbf{n} \in \overline{A}$ such that $\mathbf{n} \notin \overline{A}_T$.

Thus, $\mathbb{N} \models \mathbf{n} \notin A$ but $T \nvdash \text{“}\mathbf{n} \notin A\text{”}$. ❑

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So, there must be some $\mathbf{n} \in \overline{A}$ such that $\mathbf{n} \notin \overline{A}_T$.

Thus, $\mathbb{N} \models \mathbf{n} \notin A$ but $T \nvdash \text{“}\mathbf{n} \notin A\text{”}$. ❑

The proof in this form is not constructive, since

$\mathbf{n}$ is not (constructively) specified.

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Kleene's Constructive Proof.

Let T be a sufficiently strong[a], sound and semi-decidable theory.

$$\{n\in\mathbb{N} \mid T\vdash \varphi_n(n)\uparrow\} \subseteq \{n\in\mathbb{N} \mid \varphi_n(n)\uparrow\}.$$

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Gödel's Proof

Gödel's Proof (for sound and definable T).

Denote the n-th Formula by $\mathcal{F}_n$ (via a Gödel coding).

$$\{n \in \mathbb{N} \mid T \vdash \neg\mathcal{F}_n(\overline{n})\} \subseteq \{n \in \mathbb{N} \mid \mathbb{N} \models \neg\mathcal{F}_n(\overline{n})\}.$$

The first set is arithmetically definable, while the second set is not!
(Tarski's Theorem: if it were by $\mathcal{F}_t(x)$ then $\mathcal{F}_t(t) \leftrightarrow \neg\mathcal{F}_t(t)$!).

The first set is definable by $\mathcal{F}_g(x)$; from $\mathcal{F}_g(x) \equiv T \vdash \neg\mathcal{F}_x(x)$ we have $\neg\mathcal{F}_g(g) \leftrightarrow T \nvdash \neg\mathcal{F}_g(g)$ (Gödel's Sentence).

So, for some sentence $\mathcal{G}$ we have $\mathcal{G} \equiv T \nvdash \mathcal{G}$ (Diagonal Lemma).

Now, (1) $\mathbb{N} \models \mathcal{G}$, since otherwise $T \vdash \mathcal{G}$, and so $\mathbb{N} \models \mathcal{G}$.

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Gödel's Proof (for sound and definable T).

Denote the n-th Formula by $\mathcal{F}_n$ (via a Gödel coding).

$$\{n\in\mathbb{N} \mid T\vdash\neg\mathcal{F}_n(\overline{n})\} \subseteq \{n\in\mathbb{N} \mid \mathbb{N}\models\neg\mathcal{F}_n(\overline{n})\}.$$

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See You Later

To Be Continued ...

- Tutorial I:
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- Tutorial II:
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- Tutorial III:
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Thank You!

Thanks to

The Participants .... For Listening · · ·

and

The Organizers — For Taking Care of Everything · · ·

SaeedSalehi.ir

Hello!

GÖDEL'S INCOMPLETENESS THEOREM: Constructivity of Its Various Proofs*

SAEED SALEHI

University of Tabriz & IPM

http://SaeedSalehi.ir/

*A Joint Work with PAYAM SERAJI.

SWAMPLANDIA 2016, Ghent University
Tutorial III: Constructivity of Proofs for Gödel's Theorem
31 May 2016

Outline

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The Proof of G. Boolos

J. Barwise, *Notices of the American Mathematical Society* 36:4 (1989) 388.
"This Month's Column"

The column also contains … a very lovely proof of Gödel's Incompleteness Theorem, probably the deepest single result about the relationship between computers and mathematics, as well as having played an important (if slightly ironic) role in the development of computers, as I have discussed earlier. I am pleased to include in this column the most straightforward proof of this result that I have ever seen.

Boolos' Proof (history)

- G. Boolos, A New Proof of the Gödel Incompleteness Theorem, *Notices of the American Mathematical Society* 36:4 (1989) 388–390.
- M. Kikuchi, A Note on Boolos' Proof of the Incompleteness Theorem, *Mathematical Logic Quarterly* 40:4 (1994) 528–532.
- D.K. Roy, The Shortest Definition of a Number in Peano Arithmetic, *Mathematical Logic Quarterly* 49:1 (2003) 83–86.
- G. Serény, Boolos-Style Proofs of Limitative Theorems *Mathematical Logic Quarterly* 50:2 (2004) 211–216.
- M. Kikuchi & T. Kurahashi & H. Sakai, On Proofs of the Incompleteness Theorems Based on Berry's Paradox by Vopěnka, Chaitin, and Boolos, *Mathematical Monthly* 58:45 (2012) 307–316.
- C.C. Leary, *A Friendly Introduction to Mathematical Logic*, Prentice Hall (1999, 1st ed.) Milne Library (2015, 2nd ed.)
- S. Hedman, *A First Course in Logic: an introduction to model theory, proof theory, computability, and complexity*, Oxford Univ. Press (2004).

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Proof.

Let Def-Len(y,z) be the formula which states that "there is a formula $\varphi(x)$ with the only free variable x and the length smaller than z such that $T\vdash\forall x[\varphi(x)\leftrightarrow x=\bar{y}]$". Let Berry$(u,v)$ denote "u is the least number not definable by a formula with length less than v", i.e., $\neg$Def-Len$(u,v)\wedge\forall y<u\,$Def-Len(y,v). If ℓ is the length of Berry(u,v) let Boolos(x) = Berry$(x,\overline{5\ell})$ and let $\mathfrak{b}$ be the least number not definable by a formula with length $<5\ell$. So, BOOLOS$(\bar{\mathfrak{b}})$ IS A TRUE FORMULA; BUT IT IS UNPROVABLE IN T. Since, otherwise, if $T\vdash$ Boolos$(\bar{\mathfrak{b}})$, then, since Berry$(\mathfrak{u},v)\wedge$Berry$(\mathfrak{w},v)\rightarrow\mathfrak{u}=\mathfrak{w}$ is provable in arithmetic (and in T), $(*)$ $T\vdash\forall x[$Boolos$(x)\leftrightarrow x=\bar{\mathfrak{b}}]$. Now on the one hand we have (i) $T\vdash\neg$Def-Len$(\bar{\mathfrak{b}},\overline{5\ell})$ and on the other hand, since Def-Len$(\bar{\mathfrak{b}},\overline{5\ell})$ is a true (Σ_1-)formula by $(*)$, T can prove it: (ii) $T\vdash$ Def-Len$(\bar{\mathfrak{b}},\overline{5\ell})$; contradiction ! ❑

Also, $\neg$Def-Len$(\bar{\mathfrak{b}},\overline{5\ell})\in\Pi_1$ is a True but (T-)Unprovable Sentence.

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Let Def-Len(y,z) be the formula which states that "there is a formula $\varphi(x)$ with the only free variable x and the length smaller than z such that $T\vdash\forall x[\varphi(x)\leftrightarrow x=\bar{y}]$". Let Berry$(u,v)$ denote "u is the least number not definable by a formula with length less than v", i.e., $\neg$Def-Len$(u,v)\wedge\forall y<u\,$Def-Len(y,v). If ℓ is the length of Berry(u,v) let Boolos(x) = Berry$(x,\overline{5\ell})$ and let $\mathfrak{b}$ be the least number not definable by a formula with length $<5\ell$. So, Boolos$(\bar{\mathfrak{b}})$ IS A TRUE FORMULA; BUT IT IS UNPROVABLE IN T. Since, otherwise, if $T\vdash$ Boolos$(\bar{\mathfrak{b}})$, then, since Berry$(\mathrm{u},v)\wedge$Berry$(\mathrm{w},v)\rightarrow\mathrm{u}=\mathrm{w}$ is provable in arithmetic (and in T), $(*)\ T\vdash\forall x[\text{Boolos}(x)\leftrightarrow x=\bar{\mathfrak{b}}]$. Now on the one hand we have (i) $T\vdash\neg$Def-Len$(\bar{\mathfrak{b}},\overline{5\ell})$ and on the other hand, since Def-Len$(\bar{\mathfrak{b}},\overline{5\ell})$ is a true (Σ_1-)formula by $(*)$, T can prove it: (ii) $T\vdash$ Def-Len$(\bar{\mathfrak{b}},\overline{5\ell})$; contradiction ! ❑

Also, $\neg$Def-Len$(\bar{\mathfrak{b}},\overline{5\ell})\in\Pi_1$ is a True but (T-)Unprovable Sentence.

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Chaitin's Proof

M.D. Davis, *What is a Computation?*, in: Mathematics Today, twelve informal essays (ed. L.A. Steen, Springer 1978) p. 265; and in: Randomness and Complexity, from Leibniz to Chaitin (ed. C.S. Calude, WS 2007) p. 110.

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Definition (Kolmogorov Complexity)

$\mathscr{K}(n) = \min\{i \mid \varphi_i(0) = n\}$. ✧

Theorem (The Main (non-Constructive) Lemma)

For any m there is some ℓ such that $\mathscr{K}(\ell) > m$, and there is no computable function f such that $\forall m : \mathscr{K}(f(m)) > m$.

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Theorem (Chaitin's Theorem)

For any sound and semi-decidable theory there are w, m such that $\mathscr{K}(w) > m$ but the theory cannot prove that.

Non-Constructive Proof.

For any such T there is some m such that $T \nvdash \mathscr{K}(\omega) > m$ for any ω.
Since, otherwise if for any m there were some ω such that $T \vdash \mathscr{K}(\omega) > m$ then, for a given m, by a proof-search algorithm one could constructively find some ω with $(T \vdash)\mathscr{K}(\omega) > m$ contradicting the non-constructivity of the Main Lemma.
For a fixed such an m, by the Main Lemma, there is some w with $\mathscr{K}(w) > m$; and of course $T \nvdash \mathscr{K}(w) > m$. ❑

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BOOLOS' Proof (again)

$$\begin{tikzcd} & \mathbb{N} \\ Obj. \arrow[ur, dashed, "\text{COMPLEXITY}"] & Proc. \arrow[l, "\text{result}"] \arrow[u, "size"'] \end{tikzcd}$$

$$\text{COMPLEXITY}(object) = \min size[\text{result}^{-1}(object)]$$

Example (Logical)

$Objects = \mathbb{N}$. Fix an Arithmetical Theory T.

(sufficiently strong—can prove all the true Σ_1-sentences)

$Processes$ = formulas variables: $x, x', x'', x''', x'''', \cdots$

$$\mathcal{L}_{\text{anguage}} = \mathcal{F}_{\text{unctions}} \cup \mathcal{R}_{elations} \cup \{\neg, \rightarrow, \forall, (,), x, '\}$$

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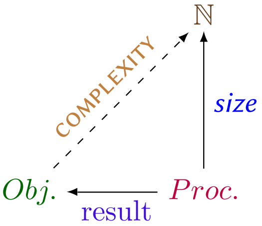

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Definition (Complexity of Definability (à la Boolos))

$\mathscr{D}_T(n) = \min\{\ell \mid \exists\varphi : \|\varphi\| = \ell \;\&\; T \vdash \forall x[\varphi(x) \leftrightarrow x = \bar{n}]\}$. ✧

Lemma (The Main Lemma on the Boolos Complexity)

For any m there is some $\hbar$ such that $\mathscr{D}_T(\hbar) > m$.

Theorem (Non-Constructivity of the Main Lemma)

There is no computable function f such that $\forall m : \mathscr{D}_T(f(m)) > m$.

Proof.

Indeed there is no such (T-)*representable* function.

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Corollary (BOOLOS — Generalized)

For any sound and semi-decidable theory T there exists some $m \in \mathbb{N}$ such that $T \nvdash \mathscr{D}_T(k) > m$ for any $k \in \mathbb{N}$.

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Boolos' Proof (again)

Theorem (Non-Constructivity)

There is no T-representable function f with $\forall m : \mathscr{D}_T\big(f(m)\big) > m.$

Proof.

If f is representable by $F(u,v)$, i.e., $T \vdash \forall x[F(\bar{m},x) \leftrightarrow x = \overline{f(m)}]$, for all $m \in \mathbb{N}$, then by the Diagonal Lemma for some formula $\mathbf{G}(x)$ we have $T \vdash \mathbf{G}(x) \leftrightarrow F(\|\mathbf{G}(x)\|, x)$. Now, for $\boldsymbol{\ell} = \|\mathbf{G}\|$, we have $T \vdash \forall x[\mathbf{G}(x) \leftrightarrow F(\bar{\boldsymbol{\ell}}, x) \leftrightarrow x = \overline{f(\boldsymbol{\ell})}]$, whence $\mathscr{D}_T\big(f(\boldsymbol{\ell})\big) \leqslant \boldsymbol{\ell}$! ❑

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> Say the m applies to n if $F([n])$ is the output of M, where $F(x)$ is the formula with Gödel number m. Let $A(x, y)$ express "applies to," and let n be the Gödel number of $-A(x, x)$. If n applies to n, the false statement $-A([n], [n])$ is the output of M, impossible; thus n does not apply to n and $-A([n], [n])$ is a truth not in the output of M.

What is concealed in this argument is the large amount of work needed to construct a suitable formula $A(x, y)$; proving the existence of the key formula $C(x, y)$ in the "New Proof" via Berry's paradox requires at least as much effort. What strikes the author as of interest in the proof via Berry's paradox is not its brevity but that it provides a different sort of reason for the incompleteness of algorithms.

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Π_1-Incompleteness Theorems

Theorem (Proofs of the Uniform Π_1-Incompleteness Theorems)

Every uniform Π_1-incompleteness is of the form

$$\text{SemiDec.}\{n\in\mathbb{N} \mid T\vdash \text{“}n\notin\mathscr{A}\text{”}\} \subsetneqq \{n\in\mathbb{N} \mid \mathbb{N}\models \text{“}n\notin\mathscr{A}\text{”}\} = \overline{\mathscr{A}}$$

for some semi-decidable and un-decidable set $\mathscr{A}$ ($\overline{\mathscr{A}}\neq$ SemiDec.).

Example (CHAITIN's Proof with $\mathfrak{C}=\{\langle a,b\rangle \mid \mathscr{K}(a)\leqslant b\}$)

By $\langle a,b\rangle\in\mathfrak{C} \iff \bigvee_{i=0}^{b}\varphi_i(0)\downarrow=a$, the set $\mathfrak{C}$ is semi-decidable, but cannot be decidable since otherwise the function $\mathscr{K}$ would be computable by $\mathscr{K}(x)=\min\{y \mid \langle x,y\rangle\in\mathfrak{C}\}-1$. ✧

Example (BOOLOS' Proof with $\mathfrak{B}=\{\langle a,b\rangle \mid \mathscr{D}_T(a)\leqslant b\}$)

Similarly, the function $\mathscr{D}_T$ is uncomputable and the set $\mathfrak{B}$ is semi-decidable and undecidable. ✧

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Non-Semi-Decidable Sets

The First Example $\overline{K}=\{n\in\mathbb{N} \mid \varphi_n(n)\uparrow\}$ Came by Diagonalizing Out.

S.C. Kleene, Origins of Recursive Function Theory,
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When Church proposed this thesis, I sat down to disprove it by diagonalizing out of the class of the λ-definable functions. But, quickly realizing that the diagonalization cannot be done effectively, I became overnight a supporter of the thesis.

Let $\mathscr{W}_n=\{x\in\mathbb{N} \mid \varphi_n(x)\downarrow\}$ be the n^{th} semi-decidable set. Every non-semidecidable set A should be different from every $\mathscr{W}_n$; there must be a function f such that $f(n)\in A\triangle\mathscr{W}_n$ for every $n\in\mathbb{N}$.

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Effectively Non-Semi-Decidable Sets

Definition (*Completely Productive*)

A set $A \subseteq \mathbb{N}$ is called *Completely Productive* if for some computable g we have $\forall x : g(x) \in A \longleftrightarrow g(x) \notin \mathscr{W}_x$. ✧

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Creative = semi-decidable + productive complement. ✧

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Remark (Not Every Non-Semidecidable is Effectively So)

There are some (uncountably many) *non–*SEMIDECIDABLE *sets which are not* (among the countable many) *effectively non–*SEMIDECIDABLE (completely productive sets). ✧

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THE SET OF ALL TRUE ARITHMETICAL FORMULAS is productive.
THE SET $K=\{n\in\mathbb{N} \mid \varphi_n(n)\downarrow\}$ is creative. ✧

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for some semi-decidable and un-decidable set $\mathscr{A}$ ($\overline{\mathscr{A}} \neq$ *SemiDec.*)

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Example (Gödel & Kleene)

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The set $\mathfrak{C} = \{\langle a, b\rangle \mid \mathscr{K}(a) \leqslant b\}$ *is not creative.*

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Theorem

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Happy Ending

G.J. Chaitin, A Century of Controversy Over the Foundations of Mathematics, *Complexity* 5:5 (2000) 12–21.

But I must say that philosophers have not picked up the ball. I think logicians hate my work, they detest it! And I'm like pornography, I'm sort of an unmentionable subject in the world of logic, because my results are so disgusting!

... the most interesting thing about the field of program-size complexity is that it has no applications, is that it proves that it cannot be applied! Because you can't calculate the size of the smallest program. But that's what's fascinating about it, because it reveals limits to what we can know. That's why program-size complexity has epistemological significance.

See You Later

THAT WAS FOR NOW ...

- Tutorial I:
  Constructive Proofs 30 May 2016
- Tutorial II:
  Gödel's Incompleteness Theorem 30 May 2016
- Tutorial III:
  Constructivity of Proofs for Gödel's Theorem 31 May 2016

Thank You!

Thanks to

The Participants .... For Listening ···

and

The Organizers — For Taking Care of Everything ···

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